MathsHUBS Resources

National Centre for Excellence in the Teaching of Mathematics

Mastery Professional Development

1 The structure of the number system

1.3 Ordering and comparing

Guidance document | Key Stage 3

Making connections

The NCETM has identified a set of six 'mathematical themes' within Key Stage 3 mathematics that bring together a group of 'core concepts'.

The first of these themes is *The structure of the number system*, which covers the following interconnected core concepts:

1.1 Place value, estimation and rounding

1.2 Properties of number

1.3 **Ordering and comparing**

1.4 Simplifying and manipulating expressions, equations and formulae

This guidance document breaks down core concept *1.3 Ordering and comparing* into three statements of knowledge, skills and understanding:

1.3.1 Work interchangeably with terminating decimals and their corresponding fractions

1.3.2 Compare and order positive and negative integers, decimals and fractions

1.3.3 Interpret and compare numbers in standard form $A \times 10^n$, $1 \leq A < 10$

Then, for each of these statements of knowledge, skills and understanding we offer a set of key ideas to help guide teacher planning.

Please note that these materials are principally for professional development purposes. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

Overview

At Key Stage 2, students worked with numbers represented in a variety of different ways and used a range of representations and manipulatives to explore the base-ten structure of integers. They encountered fractions, decimals and negative numbers, developed ways of comparing and ordering numbers and began to recognise common equivalences. As a result, students should be beginning to form a rich, connected view of the number system.

In their Key Stage 2 work on ordering numbers, students will have appreciated one aspect of the infinity of numbers (sometimes called 'unbounded infinity') – namely, that given any number, there will always be a larger or smaller number that can be placed on the number line. An important aspect of infinity that students may not meet until Key Stage 3 is that of 'bounded infinity' – namely, that given any two numbers, there will always be another number (i.e. greater than the smaller number and smaller than the larger number) that can be placed between them on a number line.

At Key Stage 3, students will further develop their understanding of the different ways that numbers can be expressed and will become more proficient in changing from one form to another. This will develop their awareness that different representations of the same number can reveal something of its structure and so can be used to compare and order numbers with ease.

Students should develop their understanding of the infinite nature of the number system and work confidently with a wide range of real numbers. For example, they should recognise that for any given pair of numbers, a third that is greater than both, smaller than both or in between the two can always be found. The process of comparing and ordering these numbers becomes easier by considering numbers expressed as decimals and fractions.

When thinking about very large and very small numbers, working with standard form notation will enable students to develop further their understanding of multiplication and division by powers of ten.

Prior learning

Before beginning to teach *Ordering and comparing* at Key Stage 3, students should already have a secure understanding of the following from previous study:

Key stage	Learning outcome
Upper Key Stage 2	• Multiply and divide whole numbers and those involving decimals by 10, 100 and 1 000 • Recognise and use thousandths and relate them to tenths, hundredths and decimal equivalents • Identify, name and write equivalent fractions of a given fraction, represented visually, including tenths and hundredths • Use common factors to simplify fractions; use common multiples to express fractions in the same denomination

	• Recognise mixed numbers and improper fractions and convert from one form to the other, and write mathematical statements > 1 as a mixed number (e.g. $\frac{2}{5}+\frac{4}{5}=\frac{6}{5}=1\frac{1}{5}$) • Read and write decimal numbers as fractions (e.g. 0.71 = $\frac{71}{100}$) • Recognise the per cent symbol (%) and understand that per cent relates to 'number of parts per hundred', and write percentages as a fraction with denominator 100, and as a decimal • Recall and use equivalences between simple fractions, decimals and percentages, including in different contexts • Use negative numbers in context and calculate intervals across zero
Key Stage 3	• 1.1.1 Understand the value of digits in decimals, measure and integers • 1.2.2 Understand integer exponents and roots • 1.3.2 Compare and order positive and negative integers, decimals and fractions **Please note:** Numerical codes refer to statements of knowledge, skills and understanding in the NCETM breakdown of Key Stage 3 mathematics.

You may find it useful to speak to your partner schools to see how the above has been covered and the language used.

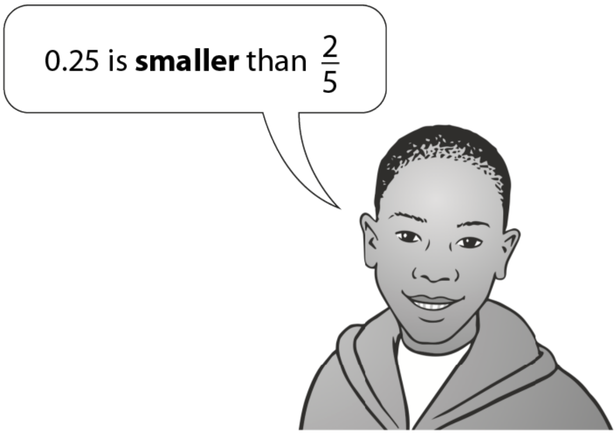
You can find further details regarding prior learning in the following segments of the NCETM primary mastery professional development materials[1]:

- Year 4: 1.23 Composition and calculation: tenths
- Year 4: 1.24 Composition and calculation: hundredths and thousandths
- Year 4: 2.13 Calculation: multiplying and dividing by 10 or 100
- Year 5: 1.27 Negative numbers: counting, comparing and calculating
- Year 5: 2.18 Using equivalence to calculate
- Year 5: 3.7 Finding equivalent fractions and simplifying fractions
- Year 6: 3.10 Linking fractions, decimals and percentages

Checking prior learning

The following activities from the NCETM primary assessment materials[2] and the Standards & Testing Agency's past mathematics papers[3] offer useful ideas for assessment, which you can use in your classes to check whether prior learning is secure:

Reference	Activity
Year 6 page 20	*Put the following numbers on a number line:* $\frac{3}{4}$, $\frac{3}{2}$, *0.5, 1.25, 3 ÷ 8, 0.125*

<table>
<tr><td>2017 Key Stage 2
Mathematics
Paper 2: reasoning
Question 20</td><td>Adam says,

Explain why he is correct.
Source: Standards & Testing Agency
Public sector information licensed under the
Open Government Licence v3.0</td></tr>
<tr><td>2018 Key Stage 2
Mathematics
Paper 2: reasoning
Question 7</td><td>Tick the two numbers that are equivalent to $\frac{1}{4}$
Tick two.
0.25 ☐
0.75 ☐
$\frac{25}{100}$ ☐
0.5 ☐
$\frac{2}{5}$ ☐
Source: Standards & Testing Agency
Public sector information licensed under the
Open Government Licence v3.0</td></tr>
<tr><td>2018 Key Stage 2
Mathematics
Paper 2: reasoning
Question 14</td><td>$\frac{6}{5}$ $\frac{3}{5}$ $\frac{3}{4}$
Write these fractions in order, starting with the smallest.
☐ ☐ ☐
smallest
Source: Standards & Testing Agency
Public sector information licensed under the
Open Government Licence v3.0</td></tr>
</table>

Key vocabulary

Term	Definition
infinite	Of a number, always bigger than any (finite) number that can be thought of. Of a sequence or set, going on forever. The set of integers is an infinite set.
standard index form (standard form)	A form in which numbers are recorded as a number between 1 and 10 multiplied by a power of ten. Example: 193 in standard index form is recorded as 1.93×10^2 and 0.193 as 1.93×10^{-1}. This form is often used as a succinct notation for very large and very small numbers.
terminating decimal	A decimal fraction that has a finite number of digits. Example: 0.125 is a terminating decimal. In contrast $\frac{1}{3}$ is a recurring decimal fraction. All terminating decimals can be expressed as fractions in which the denominator is a multiple of 2 or 5.

Collaborative planning

Below we break down each of the three statements within *Ordering and comparing* into a set of key ideas to support more detailed discussion and planning within your department. You may choose to break them down differently depending on the needs of your students and timetabling; however, we hope that our suggestions help you and your colleagues to focus your teaching on the key points and avoid conflating too many ideas.

Please note: We make no suggestion that each key idea represents a lesson. Rather, the 'fine-grained' distinctions we offer are intended to help you think about the learning journey irrespective of the number of lessons taught. Not all key ideas are equal in length and the amount of classroom time required for them to be mastered will vary, but each is a noteworthy contribution to the statement of knowledge, skills and understanding with which it is associated.

The following letters draw attention to particular features:

D Suggested opportunities for **deepening** students' understanding through encouraging mathematical thinking.

L Examples of shared use of **language** that can help students to understand the structure of the mathematics. For example, sentences that all students might say together and be encouraged to use individually in their talk and their thinking to support their understanding (for example, *'The smaller the denominator, the bigger the fraction.'*).

R Suggestions for use of **representations** that support students in developing conceptual understanding as well as procedural fluency.

V Examples of the use of **variation** to draw students' attention to the important points and help them to see the mathematical structures and relationships.

PD Suggestions of questions and prompts that you can use to support a **professional development** session.

For selected key ideas, marked with an asterisk (*), we exemplify the common difficulties and misconceptions that students might have and include elements of what teaching for mastery may look like. We provide examples of possible student tasks and teaching approaches, together with suggestions and prompts to support professional development and collaborative planning. You can find these at the end of the set of key ideas.

Key ideas

1.3.1 Work interchangeably with terminating decimals and their corresponding fractions

Students worked with fractions and decimals at Key Stage 2 and should be able to recall and use equivalences between simple fractions, decimals and percentages. They should have a good grasp of decimals up to two decimal places and be able to write decimals as fractions with a denominator of 100.

At Key Stage 2, students should have developed an awareness that any number can be expressed in a variety of ways that reveal the base-ten structure. Students should not simply name the place-value column headings. Rather, they should recognise that, for example, the number 2 437 consists of 24 hundreds or 243 tens.

At Key Stage 3, students begin to recognise decimals such as 0.13 not just as one tenth and three hundredths, but also as 13 hundredths, which leads to its expression as a single fraction. An awareness that a fraction represents a division is crucial at this stage, as this will allow a deep understanding of the process of changing fractions to decimals.

It is important during Key Stage 3 that students become increasingly fluent when converting between fractions and decimals. This will contribute to a strong sense of number, support flexibility in calculation and students' ability to make sensible choices as to when to work mentally, use a written method or use a calculator.

1.3.1.1 Understand that 1 can be written in the form $\frac{n}{n}$ (where n is any integer) and vice versa

1.3.1.2 Understand that fractions of the form $\frac{a}{b}$, where $a > b$, are greater than one and use this awareness to convert between improper fractions and mixed numbers

1.3.1.3* Understand that a fraction represents a division and that performing that division results in an equivalent decimal

1.3.1.4 Appreciate that any terminating decimal can be written as a fraction with a denominator of the form 10^n (e.g. $0.56 = \frac{56}{100}, \frac{560}{1000}$)

1.3.1.5* Understand the process of simplifying fractions through dividing both numerator and denominator by common factors

1.3.1.6 Know how to convert from fractions to decimals and back again using the converter key on a calculator

1.3.1.7 Know how to enter fractions as divisions on a calculator and understand the limitations of the decimal representation that results

1.3.2 Compare and order positive and negative integers, decimals and fractions

At Key Stage 2, students developed ways of solving problems extending beyond positive integers. They encountered negative numbers in simple number problems and real-life contexts. At Key Stage 3, students should continue to develop their understanding of negative numbers and potentially re-evaluate their understanding of 'smaller' and 'bigger' in negative contexts.

Students also worked with fractions and decimals at Key Stage 2, recognising common fractions and putting fractions in order of size. The focus now will be on developing students' methods for ordering fractions to include converting between equivalent fractions and decimals as appropriate.

Throughout this set of key ideas, students should continue to develop their understanding of how numbers can be represented differently. They should be able to apply different techniques to compare and order numbers in a variety of different contexts and have an appreciation of magnitude. For example, if students know that $0.6 > \frac{1}{2}$ and $\frac{3}{7} < \frac{1}{2}$, they should be able to deduce that $0.6 > \frac{3}{7}$ without resorting to converting to a common format. Such work will support students in being able to find a number in between any other two given numbers (whether two decimals, two fractions or one fraction and one decimal).

1.3.2.1 Compare negative integers using $<$ and $>$

1.3.2.2 Compare decimals using $<$ and $>$

1.3.2.3 Compare and order fractions by converting to decimals

1.3.2.4 Compare and order fractions by converting to fractions with a common denominator

1.3.2.5 Order a variety of positive and negative fractions and decimals using appropriate methods of conversion and recognising when conversion to a common format is not required

1.3.2.6 Appreciate that, for any two numbers there is always another number in between them

1.3.3 Interpret and compare numbers in standard form $A \times 10^n$, $1 \leq A < 10$

Students should have had considerable experience at Key Stage 2 of reading, writing and comparing numbers. They should have developed an understanding of place value and the value of digits in both whole and decimal numbers.

At Key Stage 3, students are introduced to standard index form (standard form). A key awareness underpinning a deep understanding of standard index form is that numbers can be written in multiple ways by considering multiplication and division by powers of ten.

For example, students will likely recognise that $2.3456 \times 100 = 23.456 \times 10 = 234.56 = 2345.6 \div 10 = 23\,456 \div 100$, however, the use of a power notation (such as 2.3456×10^2) to represent this will be new.

As students develop their understanding that index notation represents powers of ten, and later, of negative powers, they should begin to appreciate that such a representation is a way of writing very large or very small numbers in an efficient manner and aids comparison.

1.3.3.1* Be able to write any integer in a range of forms, e.g. $53 = 5.3 \times 10$, $530 \times \frac{1}{10}$, $5\,300 \times 0.01$

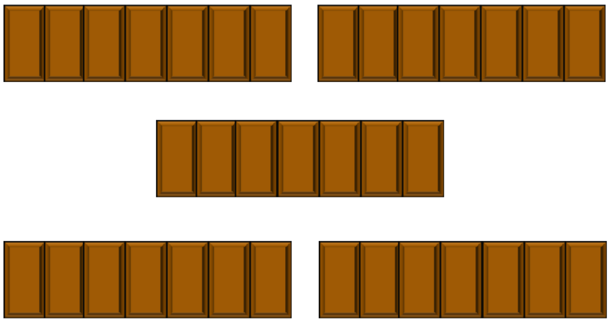
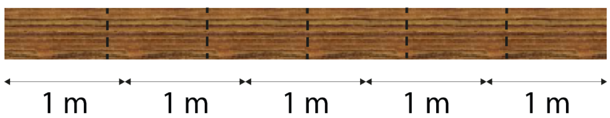
1.3.3.2 Understand that very large numbers can be written in the form $A \times 10^n$, (where $1 \leq A < 10$) and appreciate the real-life contexts where this format is usefully used

1.3.3.3 Understand that very small numbers can be written in the form $A \times 10^{-n}$, (where $1 \leq A < 10$) and appreciate the real-life contexts where this format is usefully used

Exemplified key ideas

1.3.1.3 Understand that a fraction represents a division and that performing that division results in an equivalent decimal

Common difficulties and misconceptions
Many students, when picturing a fraction such as three-fifths, will imagine a single object split into five equal parts with three of them selected. This image of a fraction is encouraged by examples such as *'A cake is cut into five equal parts. Three students each take one part. How much of the cake is eaten?'* In this interpretation both the numerator and denominator represent the same 'unit' (slices of a cake). While this image, sometimes identified as a *part–whole* construct (Behr, M., Harel, G., Post, T. R., & Lesh, R. 1992) is accurate, it is incomplete, and a second understanding of fractions (the *quotient* construct) should also be considered. In this interpretation of the fraction notation, the numerator and denominator represent different units. For example, in the question '*Three cakes are shared equally between five children. How much cake does each child eat?',* the numerator represents the number of cakes while the denominator represents the number of children. Nunes and Bryant (2009) state that most children are introduced to fractions through the part–whole model and have less experience of fractions as a quotient. They also suggest that, although the differences between these models are subtle, it is a 'crucial distinction' with at least four implications for students' understanding of fractions. Here is a summary: • The understanding of improper fractions may be easier when using the quotient construct: five cakes shared between three children is an easier way to understand five-thirds than one child eating five-thirds of a cake. • Students understand that the way in which a quantity is partitioned doesn't matter, as long as the sharing is equal. For example, if five cakes are to be shared among three children then each cake doesn't **have** to be cut into five equal parts, with each child getting three of them. • It is suggested that ordering fractions may be easier when using the quotient construct. It is likely to be easier to reason about which is larger, three-fifths or three-sixths, when considered as three cakes shared between five people or three cakes being shared between six people. • Understanding the equivalence of fractions may also be easier using the quotient construct as students may be able to reason that doubling the number of cakes and the number of children won't affect the amount of cake any child eats. When considering fractions as division, the students might find it challenging that the fraction is both the process needed to calculate the answer and the answer itself. When sharing three cakes between five people, the calculation used is $3 \div 5$ and the answer is three-fifths. This is an example of a 'procept', a term coined by Grey and Tall (1994). In their article discussing a 'proceptual view of simple arithmetic', the following quotation from Wiliam P. Thurston, a Fields Medallist, is offered. *I remember as a child, in fifth grade, coming to the amazing (to me) realization that the answer to 134 divided by 29 is 134/29 (and so forth). What a tremendous labor-saving device! To me, '134 divided by 29' meant a certain tedious chore, while 134/29 was an object with no implicit work. I went excitedly to my father to explain my major discovery. He told me that of course this is so, a/b and a divided by b are just synonyms. To him it was just a small variation in notation.* In addition to this, students who may not fully understand decimal and fraction notation often try to convert fractions to decimals by replacing the fraction bar with a decimal point (e.g. writing five-thirds as 5.3). The use of representations such as shading a hundred square may help to overcome this.

What students need to understand	Guidance, discussion points and prompts
Fraction notation represents both a division and the result of that division. *Example 1:*  a) *A chocolate bar has seven equal sections like this. Mark eats five sections of the bar. Shade this on the diagram. What fraction of the bar has Mark eaten?* b) *Five chocolate bars are shared equally between seven children.*  *Shade the diagram to show how this sharing might happen. What fraction of a bar does each child get?*	**R** Here the representation is used to allow an opportunity to explore the difference between partitioning and finding the quotient. In part b), students might be asked to find different ways to share the bars (for example, how can the bars be shared with the fewest 'breaks' of the chocolate) and this would allow for more reasoning around the connection between division and the resulting fraction of $\frac{5}{7}$.
Example 2: *A wooden plank is five metres long.* *It is to be cut into six equal shelves like this.*  a) *Write down the calculation you would use to work out how long each of the shelves will be.* b) *Fiona says, 'Each of the shelves looks like it's about three-quarters of a metre long.' How accurate is Fiona's estimate?* c) *What fraction of a metre is the length of each shelf?*	As in *Example 1*, the common understanding of division as equal sharing is used in a situation where the result is written as a fraction. Key to this example is the students' awareness of the equivalence of the calculation 5 ÷ 6 and the resulting quotient $\frac{5}{6}$. **R** The bar model representation here is used to bridge two different ways of considering fractions. The 'plank' can be considered as both one object being cut into equal pieces (the part–whole) and as a group of five lengths, each one metre long (suggesting the quotient model). The move from discrete objects in *Example 1*, towards a continuous measure is further developed in *Example 3*.

Example 3:

a) Mark the number 3 on this number line.

b) Mark the number $\frac{3}{7}$ on this number line.

What's the same and what's different about parts a) and b)?

R Again, the representation, in this case the number line, is used to give a context for two different ways to interpret a fraction.

Although placing a 3 on the 0 to 7 number line, and identifying $\frac{3}{7}$ on the 0 to 1 number line are mathematically the same (they are both $\frac{3}{7}$ of the way along the identified section of the line), the students may interpret part a) as connected with division, as they are sharing the 7 into seven equal parts in order to identify where 3 goes, and part b) as being about fractions. This task is intended to give a context to make the common language and structure of division and fractions explicit.

When working through *Examples 1–3* it is important that the students have an opportunity to verbalise and discuss their thinking, sharing their understanding and hearing alternative perspectives on each situation, so they become aware of different ways of visualising and interpreting contexts that are mathematically identical.

PD Consider the above paragraph. To what extent do you agree with the importance of hearing other perspectives on the same mathematical idea?

- Might it lead to students being confused?
- What would you do in your classroom to raise students' awareness of the different interpretations of fractions discussed in the *Common difficulties and misconceptions* section above?

Example 4:

Calculate 153 ÷ 617. Give your answer as a fraction.

PD Here, an alternative might be to reverse the task, for example ask students to write a division that gives the answer $\frac{153}{617}$. This might be a part of a more extended sequence of questions such as:

Write a division question with the answer $\frac{3}{7}$.

Now write a different division where the answer is still $\frac{3}{7}$.

And another ... And another ...

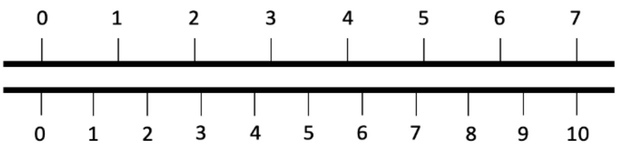
- Do you agree that the '*and another*' prompt has the same focus as the prompt used in *Example 4*?
- Which is more accessible for students in your class? When might you use one or the other?

The '*and another*' question sequence is described by Bills, Bills, Watson and Mason (2004), who stress the importance of asking students to describe how they created their examples, so they are able to generalise. How might you support students to generalise from the single example shown in *Example 4*?

Use that understanding to write a fraction as an equivalent decimal.

Example 5:

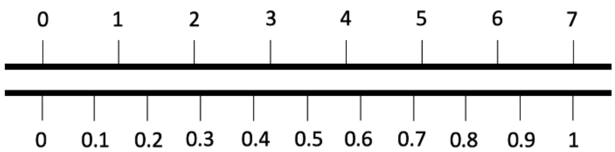
Use this double number line to estimate the value of $\frac{3}{7}$ *as a decimal.*



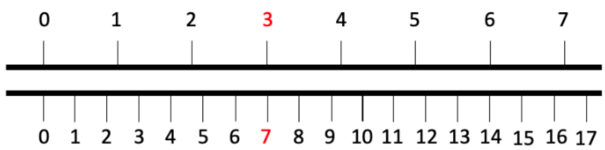
D *Example 5* challenges students to connect their understanding of decimals and equivalent fractions. Students will know that decimals can be written as tenths, hundredths and so on, but the identifying the equivalence in this context also draws on their awareness of a fraction as a division (as in *Example 3* above).

PD To what extent do you feel that this double number line adds unnecessary barriers to students' understanding?

Would the use of a decimal scale on the lower number line make the equivalence more transparent? For example:



Consider the situation where the 3 and the 7 are aligned: [1]



Is it still possible to estimate $\frac{3}{7}$ as a decimal?

[1] This situation was suggested by @profsmudge on Twitter.

1.3.1.5 Understand the process of simplifying fractions through dividing both numerator and denominator by common factors

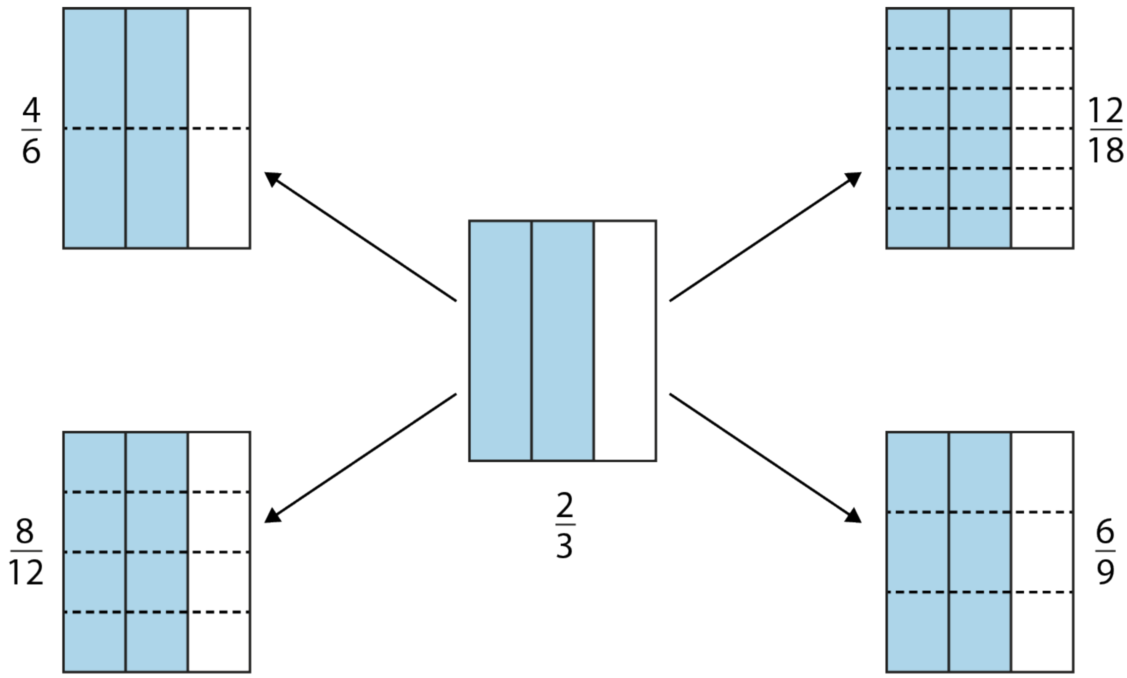
Common difficulties and misconceptions

When simplifying, or 'cancelling', fractions, students may know to look for a common factor by which to divide both the numerator and denominator. However, if this has been learnt as a procedure, with no understanding of the reason behind it, a number of misconceptions may arise. Students may, for example, think that since a division has occurred, the size of the fraction has changed. Students may also think that a 'cancelled down' fraction can be obtained by subtracting from both the numerator and the denominator, rather than by dividing.

Students should understand that the process of cancelling is the inverse of the process for obtaining equivalent fractions. It involves the scaling down of both the numerator and the denominator and, therefore, maintaining the same (multiplicative) relationship between them.

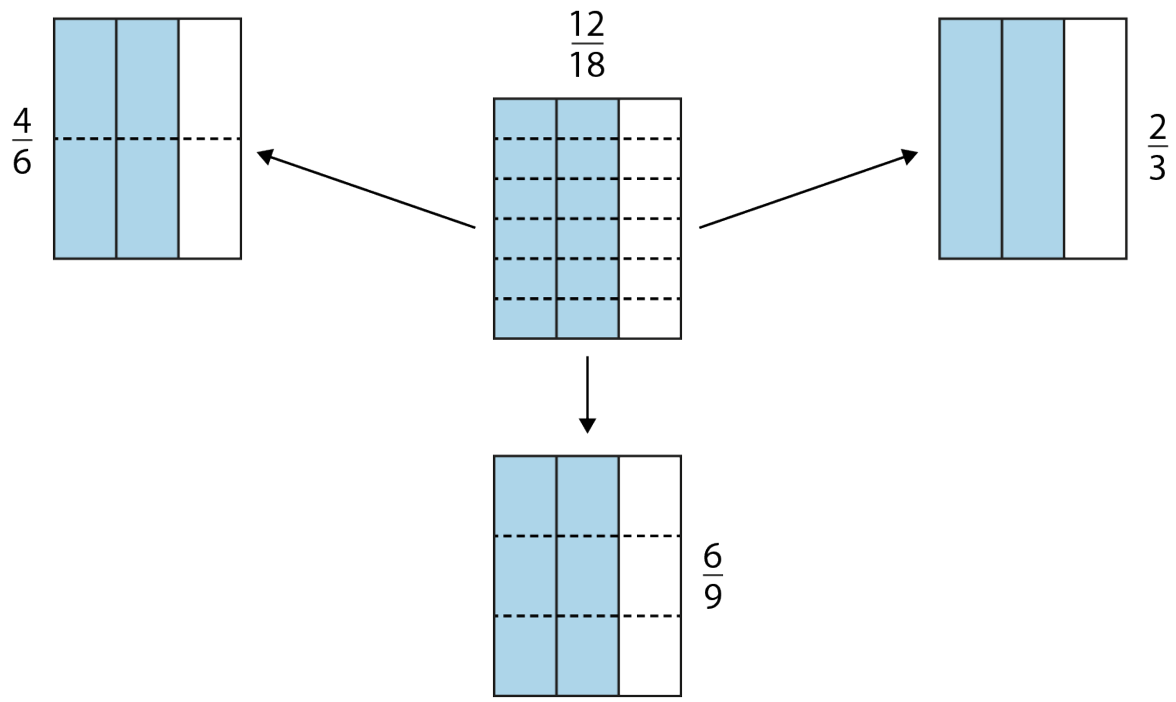
R The use of diagrams that reveal the structure of the mathematics will be important to support students' deep understanding of the concept and develop their fluency with the procedure.

For example, by sub-dividing we can create fractions equivalent to $\frac{2}{3}$:



Note what has happened to the number of blue sections and the number of white sections each time, and how this translates into the change of numerator and denominator.

By removing some sub-divisions, we can create fractions equivalent to $\frac{12}{18}$:



Note what has happened to the number of blue sections and the number of white sections each time, and how this translates into the change of numerator and denominator.

Diagrams such as the ones above will help students to appreciate that, as the denominator is halved (or divided by three, four, five, etc.) then the numerator must also be halved (or divided by three, four, five, etc.) to retain equality. However, it is important not to create the misconception that one always has to multiply by two, then by three, then by four, etc. Students need to appreciate that multiplying the numerator and denominator by the same integer yields a fraction equivalent to the original. Conversely, dividing the numerator and the denominator by the same common factor obtains an equivalent fraction.

As students work more symbolically, they may recognise common factors such as two, five and ten within fractions but may not check rigorously enough to arrive at a fraction in its simplest form. For example, students may simplify $\frac{12}{42}$ to $\frac{6}{21}$ but not cancel further by three to arrive at $\frac{2}{7}$.

L Students will have been introduced to fractions in several ways, so illustrating fractions as representations is important. When students' experience of fractions is solely through the symbolic representation, then the language of 'two over three' can dominate. This does not support students' understanding. Using diagrams, such as the ones above, can support and encourage use of the term 'out of', as in 'two out of three'.

What students need to understand

Understand how the numerator and the denominator are linked in a family of equivalent fractions.

Example 1:

Two lines on this multiplication grid have been highlighted:

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80

Discuss the patterns you see in the numbers on these two highlighted lines.

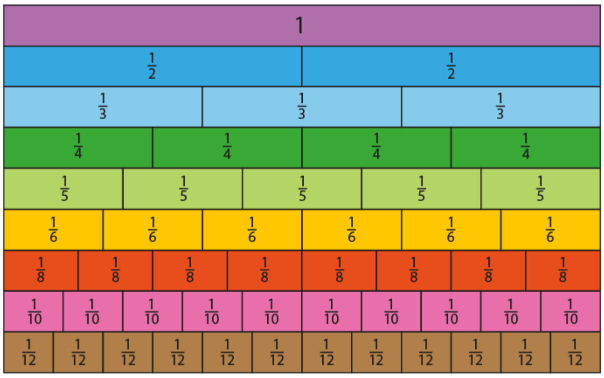
a) *What do you notice about the two sets of numbers; how are they related?*

b) *What can you say about the fractions that are formed by taking a number on the upper highlighted line as the numerator and the number below it on the lower highlighted line as the denominator?*

c) *Highlight two different lines on the multiplication grid. What fractions are revealed and what do you know about them?*

Guidance, discussion points and prompts

R Students should be familiar with finding equivalent fractions using a fraction wall like this:



For example, students should be able to use the fraction wall to find equivalent fractions to $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc. However, such activities may not allow students to see the multiplicative connection between the numerator and the denominator and appreciate that this will necessarily be the same for any other equivalent fraction.

For example, in the multiplication grid in *Example 1*, $\frac{3}{5}$ is equivalent to $\frac{6}{10}$, $\frac{9}{15}$, $\frac{12}{20}$, $\frac{15}{25}$, … Draw students' attention to the fact that, in each case, the numerator is three-fifths of the denominator.

Use of the grid in this way will support students in seeing, for example, $\frac{18}{30}$ as $\frac{3\times6}{5\times6}$, and therefore equivalent to $\frac{3}{5}$.

D Where appropriate, draw students' attention to non-integer examples, such as $\frac{2.5}{10}$.

Students should recognise that these can be expressed as proper fractions through multiplying by a convenient number. Note, multiplying a terminating decimal by ten or a power of ten will always produce integers, but not necessarily a fraction in its simplest form, e.g. $\frac{12.5}{20} = \frac{125}{200}$.

Recognise fractions in their simplest form. *Example 2:* *Is each fraction in its simplest form? Explain your reasoning.* *a)* $\frac{8}{18}$ *e)* $\frac{12}{18}$ *b)* $\frac{9}{18}$ *f)* $\frac{13}{18}$ *c)* $\frac{10}{18}$ *g)* $\frac{14}{18}$ *d)* $\frac{11}{18}$ *h)* $\frac{15}{18}$	Once students are fluent with generating equivalent fractions by multiplying both numerator and denominator by any integer, their attention can be shifted to noticing whether any common factors can be removed from certain fractions, as in *Example 2*. **V** Notice how the denominator is kept constant so that students can more easily focus on the idea of a common factor without being distracted by having to find all the factors of a different denominator each time. **L** It will be important for students to articulate how they know when a fraction is in its simplest form. Your questioning should encourage the use of some standard language, such as '… *because the numerator and the denominator have no common factors (other than one)*'.
Find a common factor and convert a fraction to its simplest form. *Example 3:* *a) Complete these equivalent fractions.* $\frac{75}{\square} = \frac{3 \times \square}{4 \times 25} = \frac{3}{\square}$ *b) Complete these equivalent fractions.* $\frac{12}{18} = \frac{\square}{9} = \frac{2}{\square}$ *c) Simplify this fraction.* $\frac{24}{36}$	Students can be given clues to help them find common factors. They can either be told to express the numerator and denominator in a certain way before dividing or they can be given one of the resulting answers so they can more easily spot the dividing factor. Additionally, exercises where students have to fill in the gaps can give a scaffold, as well as draw attention to the fact that there are many equivalent fractions and not just the simplest form. **V** In *Example 3*, part a) has been designed to highlight the highest common factor of the numerator and denominator (25). It will be important to draw students' attention to this so they understand that dividing by the highest common factor results in a fraction in its simplest form. Part b) can be used to usefully prompt discussion about what the highest common factor of 12 and 18 is, and how this is related to the interim divisions that were made:

	÷ 6 ÷ 2 ÷ 3 $\frac{12}{18} = \frac{\boxed{6}}{9} = \frac{2}{\boxed{3}}$ Part c) provides an opportunity to discuss different methods for reducing the fraction to its simplest form. It will be important to compare and contrast multiple solutions in order to arrive at a conclusion about what the most efficient method is (i.e. identifying that 12 is the highest common factor). **PD** Consider to what extent a formal explanation is required of students. Will it be acceptable for students to understand that they are dividing both the numerator and denominator by a common factor or should students be able to explain that the reason they can do this is because, in effect, they are removing the factor $\frac{k}{k}$ (i.e. one)?
Connect fractions with division and appreciate that the concept of equivalent fractions can be used to generate equivalent divisions. *Example 4:* *Explain why 8 ÷ 2, 80 ÷ 20 and 800 ÷ 200 all give the same result.*	A significant idea, which is not always grasped in the study of fractions at Key Stage 2, and which is crucial to developments at Key Stage 3, is to conceptualise fractions as a division, i.e. $\frac{3}{4} = 3 \div 4$. Working with integers offers a more familiar context to begin this work. **R** You could ask students to give several reasons (including drawing diagrams) to explain why the answers to *Example 4* are all four.
Example 5: *Express the calculation 1 300 ÷ 70 as an equivalent but simpler calculation.*	**D** You could challenge students to explain why this procedure works: $\frac{1300}{70} = \frac{130\cancel{0}}{7\cancel{0}} = \frac{130}{7}$

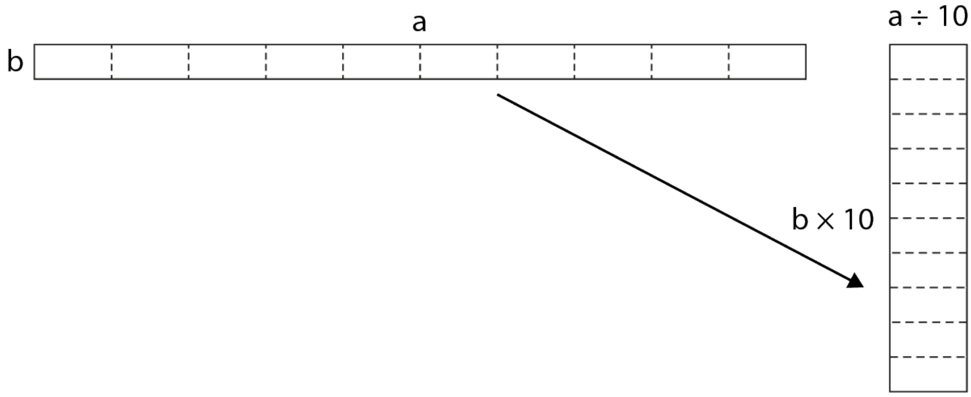
Example 6: a) *Find:* (i) $320 \div 8$ (vi) $320 \div 0.8$ (ii) $320 \div 80$ (vii) $3\,200 \div 8$ (iii) $32\,000 \div 80$ (viii) $0.32 \div 0.08$ (iv) $32 \div 8$ (ix) $0.32 \div 80$ (v) $32 \div 0.8$ b) *What is the same and what is different in each of these calculations?*	*Example 6* enables students to explore multiplying or dividing both the divisor and the dividend by powers of ten. **V** Draw attention to the connections between parts (i), (ii) and (iii): • In parts (i) and (ii), the divisor has been multiplied by 10; note what happens to the answer. (Similarly, in parts (iv) and (v) the divisor has been divided by 10.) • In parts (ii) and (iii), the dividend has been multiplied by 100. • In parts (vi) and (vii), both the dividend and the divisor have been multiplied by 10. How does this affect the answer? Use what has been learnt to find equivalent divisions to solve parts (viii) and (ix) more easily. **PD** How might you use *Example 6* as a basis for discussion of strategies for calculating (by hand) divisions where the divisor is a decimal? **D** *Example 6* might also provide a useful opportunity to ask students to think about when dividing and multiplying by ten results in a zero being removed or added, and when this cannot be done. This will prompt students to recognise that it is the digits that move under such operations.

1.3.3.1 Be able to write any integer in a range of forms, e.g. $53 = 5.3 \times 10$, $530 \times \frac{1}{10}$, $5\,300 \times 0.01$

Common difficulties and misconceptions

These equivalent forms (e.g. $53 = 5.3 \times 10$, $530 \times \frac{1}{10}$, $5\,300 \times 0.01$) are based on the key idea that when one number in a product is multiplied (or divided) by n, then, to maintain the same product, the other number must be divided (or multiplied) by n. It is important that students do not learn such manipulations by rote, without any understanding of the mathematical structures involved.

R A useful representation to reveal these structures is an area model. This can be used to show how, when one dimension (factor) is reduced by one tenth, the area is maintained by making the other dimension (factor) ten times longer:



This gives rise to equivalent products such as:

$53 \times 10 = 5.3 \times 100 = 0.53 \times 1\,000 = 530 \times 1 = 5\,300 \times 0.1$, etc.

This can be generalised to a multiplication (and corresponding division) by any scale factor, e.g. $40 \times 2.5 = 10 \times 10$ (by dividing and multiplying by four). Students should appreciate that this gives rise to other useful calculation strategies.

What students need to understand

Understand that $a \times b = (a \div n) \times (b \times n)$.

Example 1:

Given that $3.2 \times 2.5 = 8$, calculate, as efficiently as possible:

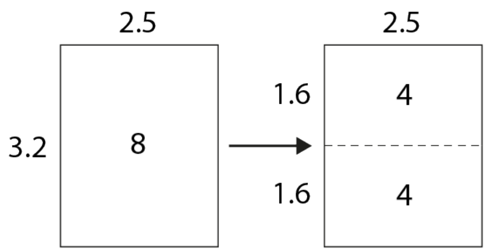
a) 1.6×2.5

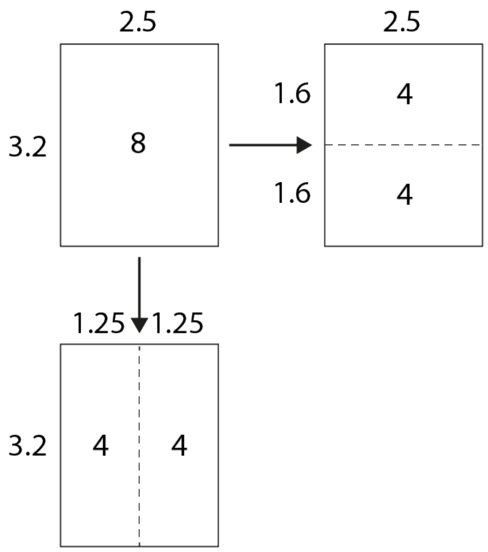
b) 3.2×1.25

Guidance, discussion points and prompts

In *Examples 1* and *2*, students should be encouraged to use the given fact to calculate (as efficiently as they can) the answers.

R You could encourage students to draw a diagram to explain why, in *Example 1*, halving one of the numbers results in halving the product. For example:



	and why the answers to parts a) and b) are the same: 
Example 2: *Given that $160 \times 0.4 = 6.4$, calculate, as efficiently as possible:* *a) 16×0.4* *b) 160×0.04*	**R** For *Example 2*, you could encourage students to construct a similar argument and diagram as in *Example 1*.
Example 3: *a) Complete this family of equations by filling in the gaps.* *(i) $0.3 \times \square = 1\,200$* *(ii) $3 \times 400 = 1\,200$* *(iii) $30 \times 40 = 1\,200$* *(iv) $300 \times \square = 1\,200$* *(v) $\square \times 0.4 = 1\,200$* *b) How would you calculate $0.0003 \times 4\,000\,000$?*	After students complete *Example 3*, you could encourage them to create their own sequences with gaps and share with other students to complete. **D** Consider prompting students to think about what happens when: • one factor is multiplied by three, or four, or … • one factor is divided by three, or four, or … • both factors are multiplied (or divided) by three, or four, or … .

Example 4: *Given that* $240 \times 0.25 = 60$ *Calculate:* *a)* 80×0.75 *b)* 20×3 *c)* 0.2×300 *d)* 0.6×100 *e)* 30×2	**L** Students should be encouraged to articulate the structures that they have noticed, using clear and precise mathematical language. For example: • *'When one factor is multiplied (or divided) by n, the product is multiplied (or divided) by n.'* • *'When one factor is multiplied (or divided) by n and the other factor is divided (or multiplied) by n, the product remains the same.'*
Understand and be able to use the index notation for powers of ten. *Example 5:* *Consider the following sequence of equalities:* $10 \times 10 \times 10 \times 10 = 10000 = 10^4$ $10 \times 10 \times 10 = 1000 = 10^3$ $10 \times 10 = 100 = 10^2$ *a) Discuss the patterns you see. Can you explain them?* *b) What would the next lines of the sequence of calculations be?*	**D** Students should be encouraged to reason about the patterns they see in *Example 5*. The following prompts could support them in doing this: • *'How is the power of ten changing from line to line? Why is this?'* • *'If you extended the pattern of powers of ten into negative numbers, what equalities would be created?'* • *'Can you give a meaning to* 10^0, 10^{-1}, 10^{-2}, *etc?'*
Use alternative ways of expressing powers of ten. *Example 6:* *Work out:* *a)* 6.7×100 *b)* $\frac{67}{10} \times 100$ *c)* 6.7×10^2 *d)* 34×0.01 *e)* $34 \times \frac{1}{100}$ *f)* $34 \times \frac{1}{10^2}$	**PD** Students should already know and understand the index notation for powers of ten. In what order do you introduce different notations for negative powers of ten in your department? In *Example 6*, we have suggested that the decimal version is introduced first (e.g. 0.01), then the fractional equivalent (e.g. $\frac{1}{100}$) then the fractional equivalent with the denominator expressed as a power of ten (e.g. $\frac{1}{10^2}$), leading to the introduction of standard form for very small numbers later. **D** In *Example 6*, discussion could focus on the structure of the calculations and not the answers. Prompts to support this might include:

	• *'Why are the answers to parts a), b) and c) equal? How has each term in the calculation changed?'* • *'In parts d), e) and f), what has changed and what has stayed the same?'* Students should be able to recognise that 0.1 is equivalent to $\frac{1}{10}$ and so when using decimals there are a number of alternative ways the decimal product can be written. For example, $80 \times \frac{1}{10} = 80 \times 0.1$, etc. Additionally, $100 = 10^2$ and $1\,000 = 10^3$, etc. Furthermore, $\frac{1}{100} = \frac{1}{10^2}$ and $\frac{1}{1000} = \frac{1}{10^3}$, etc. **D** You could ask students to find as many ways as they can to write a number, such as 670, using decimals, fractions with denominators that are powers of ten and negative powers of ten.

Weblinks

1 NCETM primary mastery professional development materials
https://www.ncetm.org.uk/resources/50639

2 NCETM primary assessment materials
https://www.ncetm.org.uk/resources/46689

3 Standards & Testing Agency past mathematics papers
https://www.gov.uk/government/collections/national-curriculum-assessments-practice-materials#key-stage-2-past-papers