MathsHUBS Resources

National Centre for Excellence in the Teaching of Mathematics

Mastery Professional Development

6 Geometry

6.3 Transforming shapes

Guidance document | Key Stage 3

Making connections

The NCETM has identified a set of six 'mathematical themes' within Key Stage 3 mathematics that bring together a group of 'core concepts'.

The sixth of these themes is *Geometry*, which covers the following interconnected core concepts:

6.1 Geometrical properties

6.2 Perimeter, area and volume

6.3 **Transforming shapes**

6.4 Constructions

This guidance document breaks down core concept *6.3 Transforming shapes* into four statements of knowledge, skills and understanding:

6.3.1 Understand and use translations

6.3.2 Understand and use rotations

6.3.3 Understand and use reflections

6.3.4 Understand and use enlargements

Then, for each of these statements of knowledge, skills and understanding we offer a set of key ideas to help guide teacher planning.

Please note that these materials are principally for professional development purposes. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

Overview

Transformations describe different ways of mapping points on a plane to other points on the plane. A way to think about, describe and classify transformations is to consider what changes and what stays the same under different transformations. This also allows for discussion about congruence and similarity.

At Key Stage 2, students will have encountered all four transformations (translation, reflection, rotation and enlargement) and learnt to distinguish between them. However, students may not have concentrated on specific features, such as the centre of rotation or the centre of enlargement.

In all four transformations, students should recognise that every element of the object (i.e. every point, every line or curve, every interior space, etc.) undergoes the same transformation and that looking at each of these elements in turn will help them to accurately construct the image.

Dynamic geometry software offers an effective tool to support the teaching of transformations. It enables students to see what happens when certain transformations are applied to objects and to make conjectures, justify and test where, for example, the image of an object under a reflection will be.

The order in which transformations have been introduced in this core concept (translation, rotation, reflection and, finally, enlargement) has been chosen to highlight how the degrees of freedom available, with regards to what can vary, are being increased. Translation maintains congruence and orientation. Rotation produces a change in orientation but maintains the 'sense' of the image – a feature which is able to change only under reflection. Translation, rotation and reflection produce congruent shapes in an increasing range of orientations and senses. Enlargement is the only transformation that does not maintain congruence (other than when the scale factor is ±1) but does maintain similarity in any orientation and sense.

In this set of key ideas, it will be useful for students to consider what's the same and what's different about an object and its image as they work on different transformations.

Prior learning

Before beginning to teach *Transforming shapes* at Key Stage 3, students should already have a secure understanding of the following from previous study:

Key stage	Learning outcome
Upper Key Stage 2	• Identify, describe and represent the position of a shape following a reflection or translation, using the appropriate language, and know that the shape has not changed • Draw and translate simple shapes on the coordinate plane, and reflect them in the axes • Solve problems involving similar shapes where the scale factor is known or can be found

You may find it useful to speak to your partner schools to see how the above has been covered and the language used.

You can find further details regarding prior learning in the following segments of the NCETM primary mastery professional development materials[1]:

- Year 4: 2.17 Structures: using measures and comparison to understand scaling
- Year 6: 2.27 Scale factors, ratio and proportional reasoning

Checking prior learning

The following activity from the NCETM primary assessment materials[2] offers a useful idea for assessment, which you can use in your classes to check whether prior learning is secure:

Reference	Activity
Year 6 page 36	*Are these statements always, sometimes or never true?* • *If a shape is reflected in an axis, it stays in the same quadrant.* • *If a shape is translated to the right and up, it stays in the same quadrant.* • *If a shape is translated to the left and down, it stays in the same quadrant.* *Explain your decisions.*

Key vocabulary

Term	Definition
centre of enlargement	Mentioned in definition for enlargement: a transformation of the plane in which lengths are multiplied whilst directions and angles are preserved. A centre and a positive scale factor are used to specify an enlargement. The scale factor is the ratio of the distance of any transformed point from the centre to its distance from the centre prior to the transformation. Any figure and its image under enlargement are similar.
centre of rotation	Mentioned in definition for rotation: in 2-dimensions, a transformation of the whole plane, which maps each point to another by rotating it by a specified angle (the angle of rotation) about a fixed point (the centre of rotation).
congruent (figures)	Two or more geometric figures are said to be congruent when they are the same in every way except their position in space. Example: Two figures, where one is a reflection of the other, are congruent since one can be transposed onto the other without changing any angle or edge length.
enlargement	A transformation of the plane in which lengths are multiplied whilst directions and angles are preserved. A centre and a positive scale factor are used to specify an enlargement. The scale factor is the ratio of the distance of any transformed point from the centre to its distance from the centre prior to the transformation. Any figure and its image under enlargement are similar.
image	When a transformation is applied to a shape, the transformed shape is called the 'image'. The original shape is called the 'object'.

object	When a transformation is applied to a shape, the original shape is called the 'object'. The transformed shape is called the 'image'.
scale factor	For two similar geometric figures, the ratio of corresponding edge lengths.
similar	Two shapes are similar if an enlargement of one will produce the other. This may be an enlargement of scale factor 1, although these shapes would then be 'congruent'. Two similar shapes do not have to share the same orientation nor the same sense.

Collaborative planning

Below we break down each of the four statements within *Transforming shapes* into a set of key ideas to support more detailed discussion and planning within your department. You may choose to break them down differently depending on the needs of your students and timetabling; however, we hope that our suggestions help you and your colleagues to focus your teaching on the key points and avoid conflating too many ideas.

Please note: We make no suggestion that each key idea represents a lesson. Rather, the 'fine-grained' distinctions we offer are intended to help you think about the learning journey irrespective of the number of lessons taught. Not all key ideas are equal in length and the amount of classroom time required for them to be mastered will vary, but each is a noteworthy contribution to the statement of knowledge, skills and understanding with which it is associated.

The following letters draw attention to particular features:

D Suggested opportunities for **deepening** students' understanding through encouraging mathematical thinking.

L Examples of shared use of **language** that can help students to understand the structure of the mathematics. For example, sentences that all students might say together and be encouraged to use individually in their talk and their thinking to support their understanding (for example, *'The smaller the denominator, the bigger the fraction.'*).

R Suggestions for use of **representations** that support students in developing conceptual understanding as well as procedural fluency.

V Examples of the use of **variation** to draw students' attention to the important points and help them to see the mathematical structures and relationships.

PD Suggestions of questions and prompts that you can use to support a **professional development** session.

For selected key ideas, marked with an asterisk (*), we exemplify the common difficulties and misconceptions that students might have and include elements of what teaching for mastery may look like. We provide examples of possible student tasks and teaching approaches, together with suggestions and prompts to support professional development and collaborative planning. You can find these at the end of the set of key ideas.

Key ideas

6.3.1 Understand and use translations

When an object undergoes a translation, the size of its angles and the lengths of its lines are maintained so that the object and image are congruent. This property is shared with both rotation and reflection, but a translation, uniquely, always maintains the orientation of the object in the image. The use of notation to record a translation may follow from a need to describe it accurately and succinctly. Initially, students are likely to use informal language as they develop their understanding of the transformation, describing, for example, a move of ‘three across’ and ‘two down’. While the formal use of vectors is part of the national curriculum Key Stage 4 programme of study, translation offers students a natural opportunity to formalise their intuitive understanding about the distinction between movement and position. As a result, formal vector notation (e.g. $\begin{pmatrix} +3 \\ -2 \end{pmatrix}$) could be introduced at Key Stage 3.

6.3.1.1 Understand the nature of a translation and appreciate what changes and what is invariant

6.3.1.2 Understand the minimum information required to describe a translation (vertical and horizontal displacement)

6.3.1.3 Translate objects from information given in a variety of forms

6.3.2 Understand and use rotations

As with translations, rotations maintain congruence but offer a further degree of change between the object and the image, since the orientation of the object is not necessarily maintained. In Key Stage 2, students will have worked with objects rotated through a half, a quarter and three quarters of a turn, and this is generalised to any angle at Key Stage 3 (specifying the size and direction of turn). In addition, more attention is paid to the centre of rotation (the one point which does not move under the rotation) and the fact that the position of the image changes with different centres of rotation, even though the orientation may not.

In the construction of examples to support students’ understanding, it will be important to vary the position of the centre of rotation to include:

- on a vertex of the object
- lying within the object
- lying outside of the object.

6.3.2.1 Understand the nature of rotations and appreciate what changes and what is invariant

6.3.2.2* Understand the minimum information required to describe a rotation (centre of rotation, size and direction of rotation)

6.3.2.3 Rotate objects using information about centre, size and direction of rotation

6.3.3 Understand and use reflections

Transforming an object by reflecting it offers the full range of possible congruent shapes, and a context in which congruence may be explored further. Reflection in lines which are neither horizontal nor vertical presents increased challenge and requires students to have a sense of where the image will be. Using a range of tools, such as dynamic geometry software, alongside pencil and paper methods, gives students a greater depth of understanding.

6.3.3.1 Understand the nature of reflections and appreciate what changes and what is invariant

6.3.3.2* Understand the minimum information required to describe a reflection (line of reflection)

6.3.3.3 Reflect objects using a range of lines of reflection (including non-vertical and non-horizontal)

6.3.4 Understand and use enlargements

Students are likely to be familiar with enlargements through their work on similar shapes in Key Stage 2. At Key Stage 3, students are introduced to the idea of a centre of enlargement and that the position of this in relation to the object affects the image's position. In this set of key ideas, students consider the range of possible outcomes with an enlargement. They should come to appreciate that enlargement is the only transformation that does not guarantee a congruent shape.

At Key Stage 3, the focus is on enlargements with a scale factor ≥ 1, but the use of dynamic geometry software offers students an opportunity to reason mathematically about the images that will result if a scale factor outside of this range is used (as it is in Key Stage 4), and to then test and refine their conjectures.

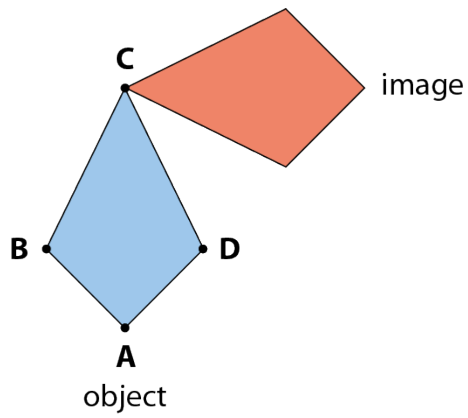
6.3.4.1 Understand the nature of enlargements and appreciate what changes and what is invariant

6.3.4.2 Understand the minimum information required to describe an enlargement (centre of enlargement and scale factor)

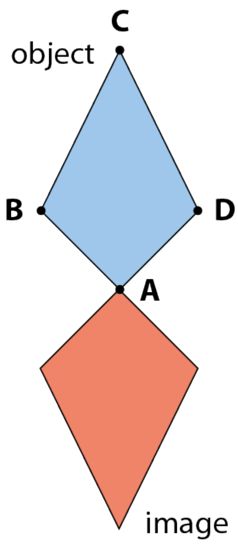
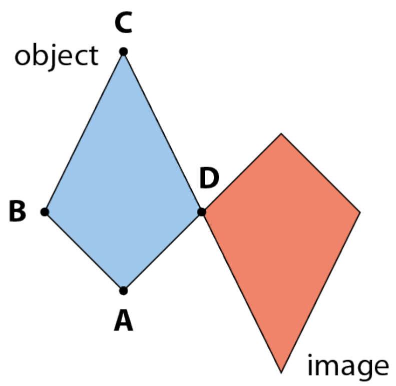
6.3.4.3 Enlarge objects using information about the centre of enlargement and scale factor

Exemplified key ideas

6.3.2.2 Understand the minimum information required to describe a rotation (centre of rotation, size and direction of rotation)

Common difficulties and misconceptions	
A rotation is arguably one of the more challenging transformations for students to visualise. You can support students in learning about rotations by giving them plenty of opportunities to experiment and become familiar with the behaviour of rotations; for example, through the use of dynamic geometry software and hands-on activities using cut-out shapes or tracing paper. Describing and recording a rotation can prove challenging for students, since it draws on their understanding of angle as a measure of turn. Research† suggests that this understanding of angle is not common in students in early Key Stage 3; rather, an angle is often viewed only as a static measure of the relationship between two lines (i.e. a measure of 'pointedness'). Rotating several key points or elements of an object to obtain the image may help to establish a better understanding. The use of dynamic geometry software enables the exploration of a range of examples.	
What students need to understand	**Guidance, discussion points and prompts**
Understand that three pieces of information are required to fully describe a rotation. *Example 1:* *The image is a 90° anticlockwise rotation of the object about point C. Describe another rotation that also transforms the object to this image.* 	Rotations require three pieces of information to be fully described: a centre of rotation, a size of rotation and a direction of rotation. *Example 1* is designed to prompt students to think about each of these pieces of information in order to find another way of describing the rotation. **PD** Later, students will be required to use their knowledge of coordinates and algebraic graphs to describe transformations. What foundations can be built to support this later learning?

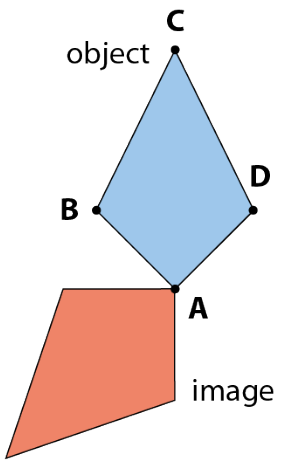
† M.C. Mitchelmore & P. White, 2000, Development of angle concepts by progressive abstraction and generalisation, *Educational Studies in Mathematics*, 41, 209–238
G. S. Close, 1982, *Children's understanding of angle at the primary/secondary transfer stage*, London, Polytechnic of the South Bank

Example 2: *The image is a rotation of the object about point A. What other information is required to fully describe this rotation?* 	In *Example 2*, students are encouraged to think about each of the key pieces of information needed to specify a rotation.
Example 3: *The image is a rotation of the object. Describe two possible rotations that transform the object to this image.* 	The ability to visualise different rotations is an important skill. In *Example 3*, students are invited to imagine rotating the object around the point D. Whole-class discussion may reveal that some students rotated the object clockwise, while others rotated it anticlockwise. **D** You could encourage students to consider the reverse transformation. That is, how would the description of the rotation differ if the object and the image were exchanged?

Use geometrical reasoning and/or equipment, such as angle measurers, to fully describe a rotation.

Example 4:

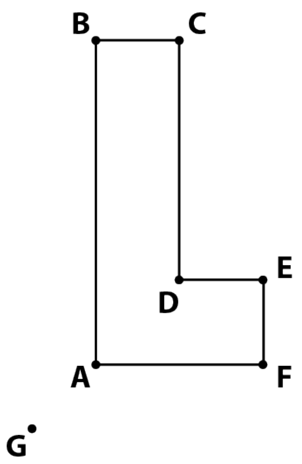
Describe two possible rotations (specifying the number of degrees turned) that transform the object to the image.



D Students may describe rotations as fractions of whole turns. *Example 4* encourages them to make connections with their knowledge of angles measured in degrees.

Understand that the position of the centre of rotation has an effect on the image's position relative to the object.

Example 5:



a) *Using tracing paper, find the position of the image if this object is rotated 90° clockwise about:*

(i) A *(iii) F*

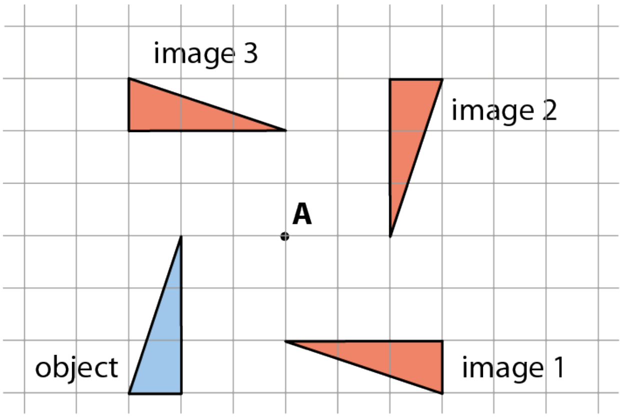
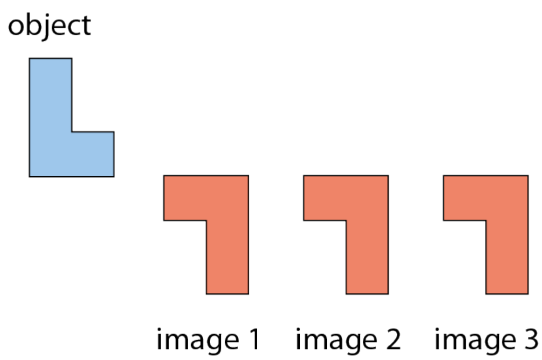
(ii) D *(iv) G*

b) *What's the same and what's different about these four images?*

V In *Example 5*, the centre of rotation is varied, but all other features of the rotation are the same. This should draw students' attention to the following key points:

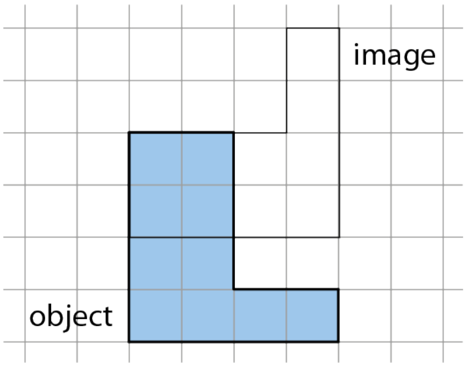
- The orientation of the image is the same wherever the centre of rotation is.
- Once one point and one line have been identified on the image, all other points and lines can be determined (because the object and the image are congruent).
- If the centre of rotation is a point on the shape, then that point does not move under the transformation.

D You could ask students to conjecture what would happen if the centre of rotation was a point inside the shape.

Example 6:  a) *What's the same and what's different about the three rotations which transform the object to images 1, 2 and 3?* b) *Describe in full each of the rotations.*	**PD** **V** In *Example 6*, the centre and direction of rotation have been kept the same while varying the angle. This is to direct students' attention to the size of the turn.
Example 7:  a) *What's the same and what's different about the three rotations which transform the object to images 1, 2 and 3?* b) *Describe in full each of the rotations.*	**V** **PD** What might students notice from keeping the size of the turn and direction of rotation constant, but varying the position of the centre of rotation?

Example 8:

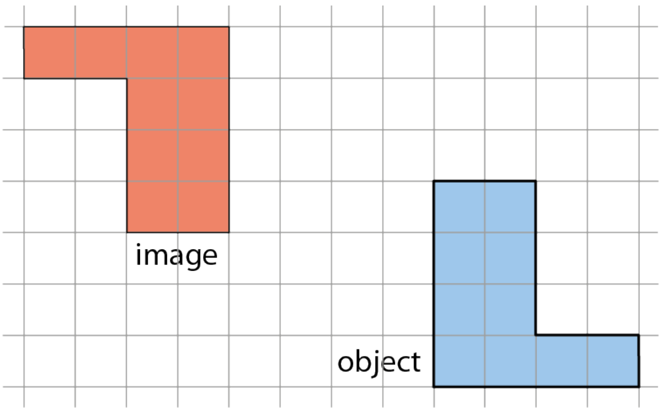
The object has been rotated 90° anticlockwise. Where is the centre of rotation?



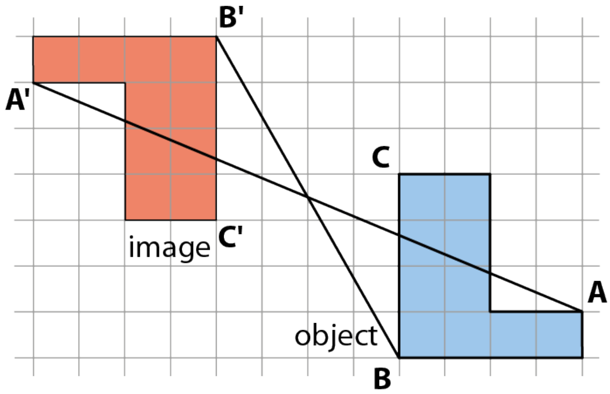
V You could encourage students to vary, for themselves, the centre of rotation and explore what happens to the position of the image. Discussion could draw out the important point that the orientation of the image will be the same for different centres if the direction and size of rotation remains the same.

Example 9:

The object has been rotated by 180°. Where is the centre of rotation?



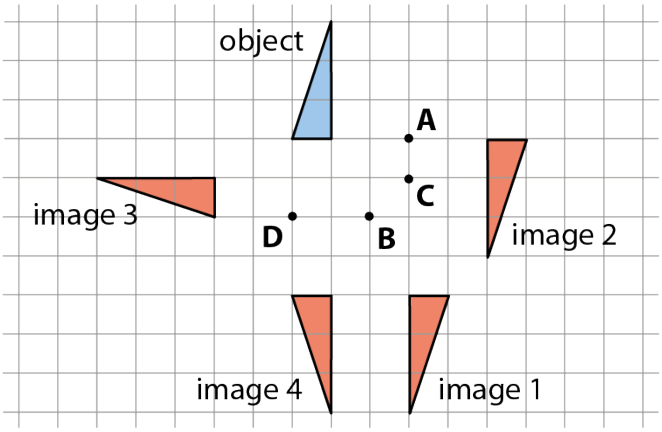
D *Example 9* provides an opportunity for students to deepen their understanding of using constructions to find the centre of rotation. By joining points on the object to corresponding points on the image, and identifying where the lines cross (as in the diagram below), the centre of rotation can be found accurately.



Recognise and fully describe a given rotation.

Example 10:

Four transformations of the blue object are shown here.



a) *Which image is not a rotation of the object?*

b) *Fully describe the transformation of the object to each image.*

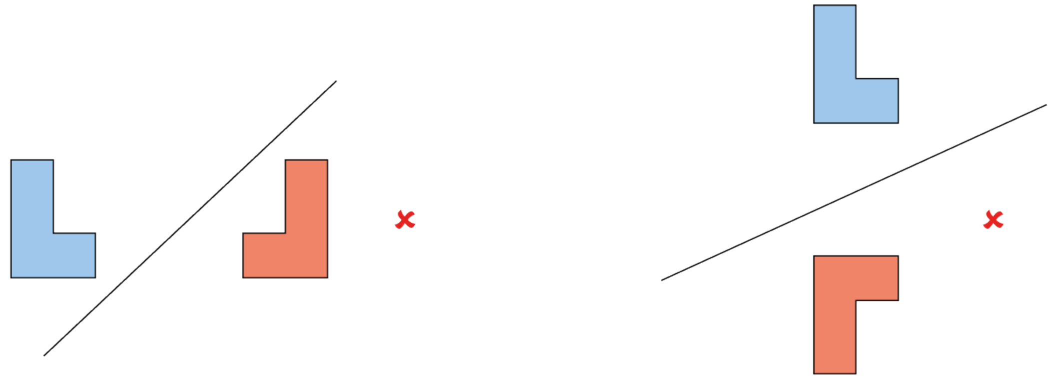
V The first part of *Example 10* draws students' attention to what a rotation *is not* by the inclusion of image 4, which is a reflection of the object in the line DB.

You could challenge students to create their own example collections of transformations, some of which are correct and others that are not, to swap with a desk partner to solve.

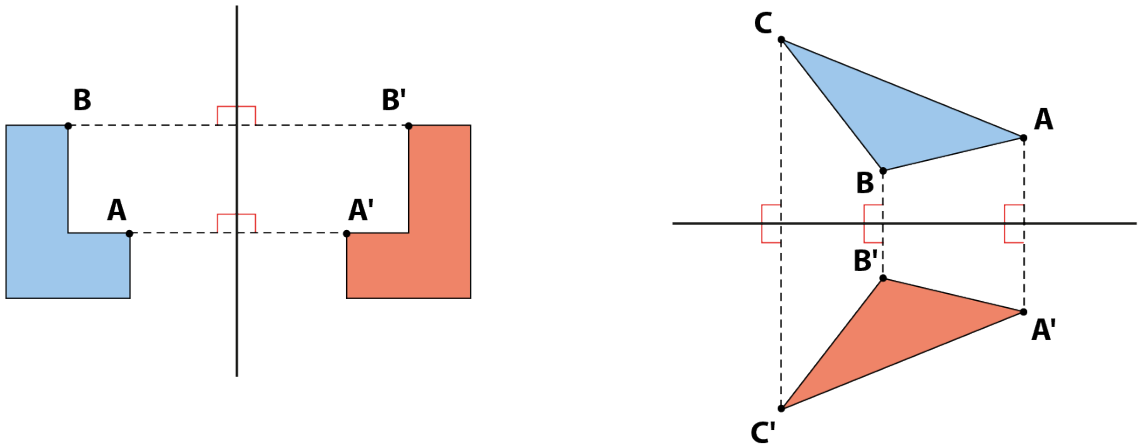
L Encourage students to use precise language to describe a rotation. For example: *'Object A is rotated 90 degrees anticlockwise around the point B.'*

6.3.3.2 Understand the minimum information required to describe a reflection (line of reflection)

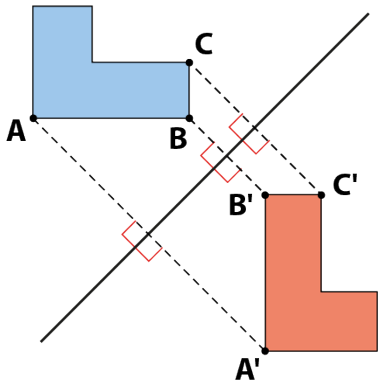
Common difficulties and misconceptions
Many students are able to intuitively reflect an object in a vertical or horizontal mirror line, but the reflection of an object in a line that is not vertical or horizontal often proves a challenge. A common misconception is that the image will remain in the same vertical (or horizontal) plane, as below:



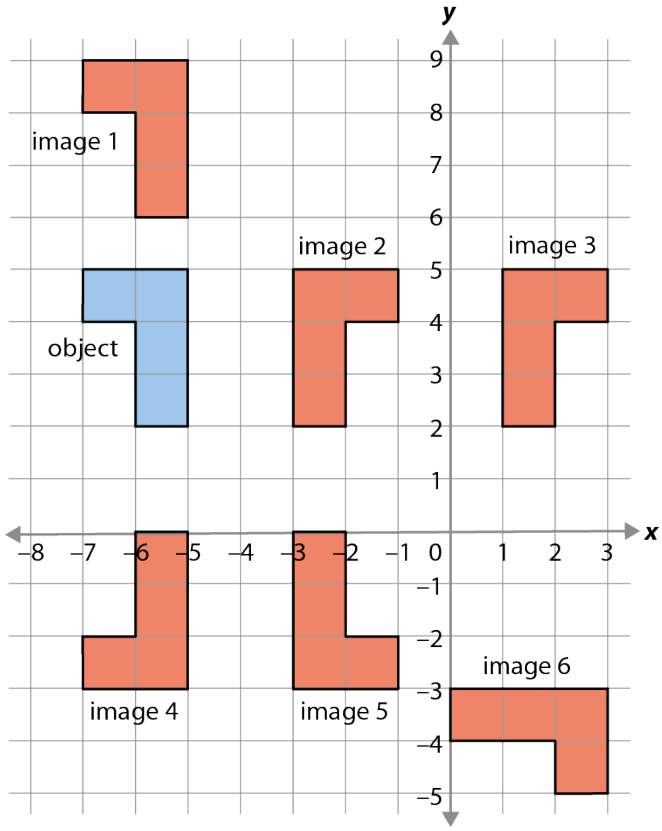
This misconception may well be due to an overuse of vertical (or horizontal) lines of reflection, so it will be important to offer students a wide range of non-standard, as well as standard, examples of lines of reflection. Furthermore, when offering students examples where the line of reflection is either vertical or horizontal, it will be important to avoid explanations that refer to reflecting ‘straight across’ or ‘straight down’. Instead, draw students’ attention to the fact that every line which joins a point on the object to its image is **perpendicular** to the line of reflection.



Making this clear when working with vertical and horizontal lines of reflection will support students in generalising the idea and help to avoid such misconceptions.



Students should be provided with experience of examples that include objects and images in any quadrant of the graph and overlapping the axes. Mirror lines should include the axes and lines with a gradient of 1 and –1, including $y = x$ and $y = -x$.

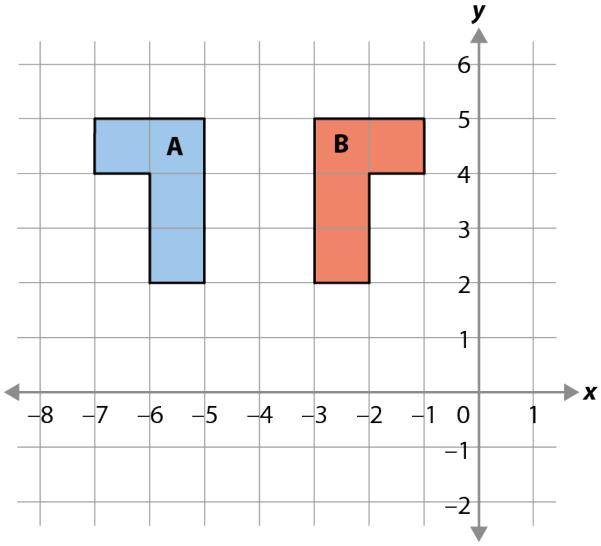
What students need to understand	**Guidance, discussion points and prompts**
Recognise transformations that are reflections. *Example 1:* *Which images are reflections of the object?* 	**V** *Example 1* shows reflections of the object in different positions and orientations. Images 2, 3, 4 and 6 are reflections. Image 5 has been chosen to raise questions around reflections that are not aligned horizontally and/or vertically with the object and should be compared with image 6 (a reflection in a line with a gradient of one). Meanwhile, image 1 has been chosen to draw attention to the change in ‘sense’ that is a necessary feature of a reflection.

Describe reflections on a coordinate grid with mirror lines that are parallel to an axis.

Example 2:

a) *Draw the mirror line to reflect shape A onto shape B.*

b) *Describe the reflection of shape A onto shape B.*

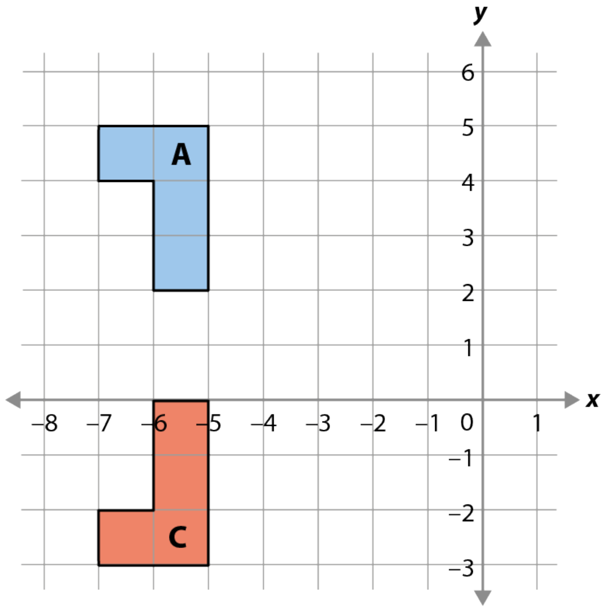


Example 2 focuses on different aspects of finding a mirror line. In part a), students are asked to identify the mirror line and draw it in, so that the focus in part b) can be on communicating their answer accurately and efficiently (this is likely to be by giving the equation of the line).

L Encourage students to use precise language to describe a reflection. For example: *'Shape A is reflected in the line $x = -4$.'*

Example 3:

Describe the transformation of shape A onto shape C.



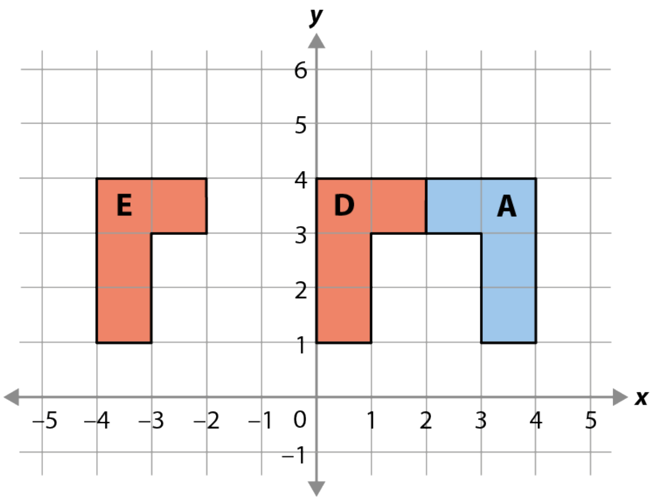
D In *Example 3*, 'transformation' is used to prompt students to think about the possibility of the transformation being something other than a reflection. Students should be encouraged to reason why it cannot be a rotation or a translation, and to think about whether it could possibly be an enlargement.

Once students have solved the problem, you might encourage them to reflect on some of the decisions they made. Prompts might include:

- *'What did you look for to identify that this was a reflection rather than a different transformation?'*
- *'Once you had spotted it was a reflection, what did you look for to identify the mirror line?'*
- *'What is the best way to describe the mirror line to someone else?'*

Example 4:

Describe the transformations of shape A onto shape D and shape E.



V *Example 4* presents two non-standard transformations to students. Image D is in contact with object A, highlighting that there does not need to be a space between the object (and image) and the mirror line. Image E is a reflection in the *y*-axis.

D There is an opportunity here for students to describe the axis of reflection of A onto E as either 'the *y*-axis' or '*x* = 0'.

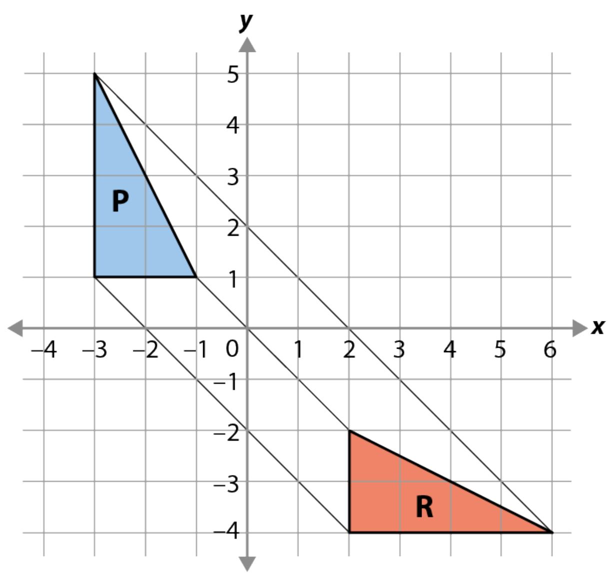
Students could consider whether objects and their images can overlap when reflected. By imagining the change in the mirror line from E to D as a dynamic change, students may be able to visualise the impact of moving the line and so consider the impact of moving the line even further right than that for image D. Once given the chance to discuss and visualise the situation, students should then test their ideas either by drawing or using dynamic geometry software.

Identify that the mirror line is perpendicular to lines joining points on the object and image.

Example 5:

In diagram a), object P has been reflected to give the image R. The vertices of P and R have been joined.

a)



V All of the representations used so far have been aligned in such a way that the angle of the mirror line has not been an explicit consideration. By changing the angle of the mirror line, greater depth of understanding about the angle between the object and line can be revealed.

Draw students' attention to the way in which lines joining equivalent points on the object and image are parallel. This insight, combined with their understanding so far, should allow them to predict the position of the mirror line.

Students should be made aware that the mirror line is **always** perpendicular to the lines joining the equivalent vertices. This can be shown effectively using dynamic geometry software.

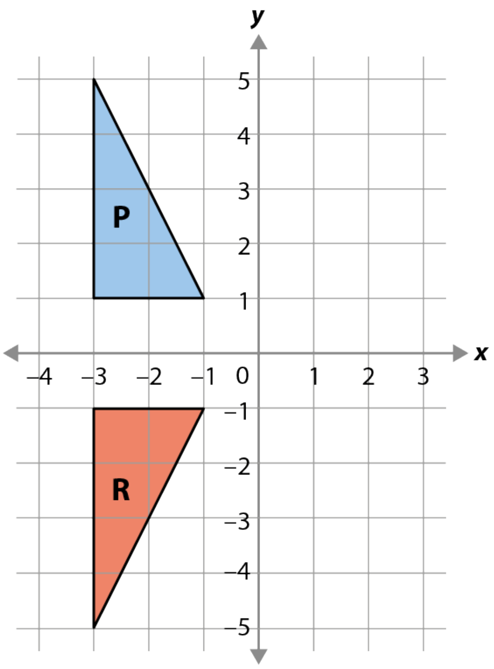
For diagrams b) and c):

(i) Join the vertices of the object, P, to the vertices of the image, R.

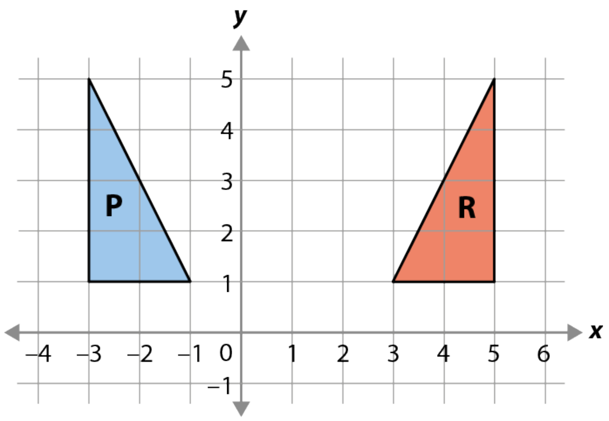
(ii) Mark the mirror line.

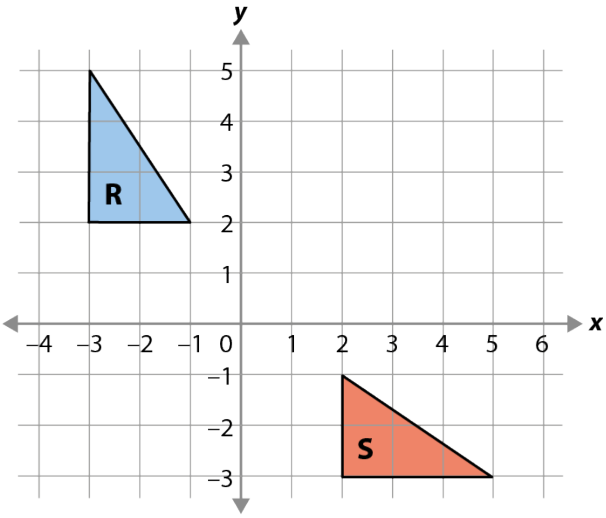
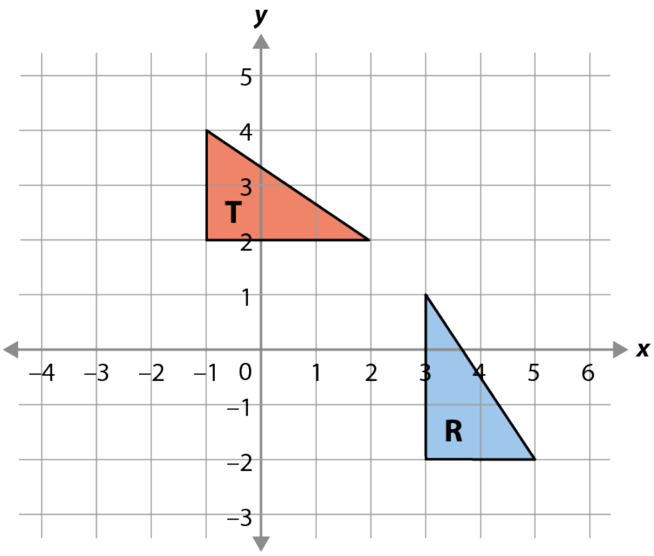
(iii) What do you notice about the angle between the mirror line and the lines joining the vertices?

b)



c)



Describe reflections on a coordinate grid with mirror lines that are of the form $y = x + c$. *Example 6:* a) *Draw the mirror line to reflect shape R onto shape S.* b) *Describe the reflection of shape R onto shape S.* 	As in *Example 2*, this question is structured to focus students' attention on identifying the mirror line, before describing it. Whether students describe the mirror line using its equation is a decision that will be driven by their prior knowledge. You might like to direct students to join the equivalent vertices in the object and the image (as in *Example 5*) before asking them to draw the mirror line. The key awareness for students is that the mirror line is perpendicular to lines joining points on the object and image.
Example 7: *Describe the transformation of shape R onto shape T.* 	**V** In *Example 7*, the object and image appear in a non-standard position – they are placed over the axes. It is important not to always present students with standard examples where mirror lines are vertical or horizontal, or, when sloping, always placed at $y = x$ or $y = -x$, etc., as this tends to promote a lack of analysis and reasoning.

Weblinks

1 NCETM primary mastery professional development materials
https://www.ncetm.org.uk/resources/50639

2 NCETM primary assessment materials
https://www.ncetm.org.uk/resources/46689