



Teaching for Mastery

Questions, tasks and activities to support assessment in KS3

Assessing Mastery in the Mathematics National Curriculum in Key Stage Three

About the authors

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Introduction

In 2015, the NCETM, working with a similar group of experts in primary mathematics, compiled a series of e-booklets to support teachers in making judgements about the degree to which their pupils were mastering mathematics. There were six booklets, one for each primary year group, and each consisting of tasks and activities that teachers could use to assess children's understanding.

This document, published in Autumn 2017, aims to provide similar support for teachers in the area of Key Stage 3 mathematics. The design closely follows that of the primary materials. For a fuller description of the NCETM's thinking and work in the field of mastery, across all school age groups, please see the mastery section of the NCETM website. <https://www.ncetm.org.uk/mastery>

The aims of the National Curriculum for mathematics, that pupils should become fluent, reason mathematically and solve problems, will be developed as a pupil journeys through their school life. Good assessment ensures that this journey is both coherent and consistent, allowing the teacher to build upon prior understanding and move the pupils forward, rather than repeating already understood concepts, or leaping forward and leaving gaps that will inevitably become exposed as the pupil tries to build on them¹.

The National Curriculum states that:

“ Mathematics is a creative and highly inter-connected discipline that has been developed over centuries, providing the solution to some of history's most intriguing problems. It is essential to everyday life, critical to science, technology and engineering, and necessary for financial literacy and most forms of employment. A high-quality mathematics education therefore provides a foundation for understanding the world, the ability to reason mathematically, an appreciation of the beauty and power of mathematics, and a sense of enjoyment and curiosity about the subject. ”

Assessment of mathematics must give pupils the opportunity to demonstrate their current capacity with these features. It should expose misconceptions and misunderstandings and provide information for the teacher about what the pupils know, understand and can do if it is to offer a full and rounded picture of the pupils' mathematics.

These materials outline the key mathematical skills and concepts within the Key Stage 3 Programme of Study and offer examples of questions that may support both teaching and assessment. The activities offered are not intended to address each and every programme of study statement in the National Curriculum. Rather, they attempt to highlight the key themes and big ideas.

¹ Hodgen and Wiliam (2006) describe 'the first principle of learning, which is to start from where the learner is, recognising that students have to reconstruct their ideas and that to merely add to those ideas an overlay of new ideas tends to lead to an understanding of mathematics as disconnected and inconsistent.' (Hodgen, J. and Wiliam, D., 2006. Mathematics inside the black box: Assessment for learning in the mathematics classroom.)

The structure of the materials

The materials are organised in a framework as outlined below.

National Curriculum POS	
Big Idea	
Mastery Contains examples of what children need to be able to do to demonstrate deep and sustainable learning	Mastery with Greater Depth Contains examples of how pupils continue to explore the mathematics by going deeper

The primary Programme of Study is split into distinct objectives for each year group and so the primary equivalents of this document target particular objectives defined by a pupil's age. As pupils move to Key Stage 3, the Programme of Study no longer defines objectives by year; rather each school is able to determine their own progression through the mathematics that is to be learned. No attempt has been made in this document to identify tasks or questions as being appropriate for a particular year group. Your department's scheme of work will describe the mathematical content that pupils should learn in line with the Key Stage 3 Programme of Study, and the questions in this document are designed to assess understanding of that content.

The National Curriculum statements have been placed in groups that exemplify the same big idea.

This (November 2017) version differs only slightly in substance from the October 2017 version previously on the NCETM website. The majority of changes are in formatting and style. The only maths-related changes are (i) on page 7, $10n$ has been changed to 10^n ; (ii) on page 19, the distance/time graph, and associated questions, have been changed, and (iii) on page 9, the wording of the second prime factorisation statement has been altered.

NUMBER

Selected National Curriculum Programme of Study statements

Pupils should be taught to:

- Understand and use place value for decimals, measures and integers of any size
- Round numbers and measures to an appropriate degree of accuracy [for examples, to a number of decimal places or significant figures]
- Use approximations through rounding to estimate answers and calculate possible resulting errors expressed using inequality notation $a < X \leq b$

The Big Ideas

This group of statements focuses on pupils' understanding of place value, estimating and rounding.

Estimation is a useful context for pupils to build their reasoning skills, explaining the approximations they have taken, the impact that these approximations will make on the outcome and to consider the margin of error in their estimation. Estimation also gives pupils an opportunity to understand that a problem may have more than one correct solution, and to compare and consider which solution is likely to be more reasonable.

Fermi Problems² offer a useful context for pupils to show their estimation, calculation and reasoning skills.

Mastery

The population in England is 53 million, rounded to the nearest million.

- What is the largest that the population could be?
- What is the smallest that the population could be?

Explain how you decided.

What is the value of the highlighted digit in 54 140 308?

- What happens to the value of that digit when 54 140 308 is multiplied by 10?
- What happens to the value of that digit when 54 140 308 is divided by 100?
- What happens to the value of that digit when 10 000 is added to 54 140 308?
- What happens to the value of that digit when 1000 is subtracted from 54 140 308?
- What if the original number is 5414.0308? What changes in your answers?

Mastery with Greater Depth

The population in England is 53 million, rounded to the nearest million

a) Jane says that the population in England can be expressed as $52.5 \text{ million} < X \leq 53.5 \text{ million}$ where X represents the number of people in the population. Explain what she means.

b) Harry says he thinks that the population is 53 254 087. Comment on the accuracy of his suggested figure.

Why might it not be possible to identify the first three places in an Olympic competition if results were taken to one decimal place? Is this true for all measures?³

²Fermi problems, named after the physicist Enrico Fermi, are problems that involve estimation and reasoned guessing about values that are not known. A famous example is to estimate the number of piano tuners in Chicago.

³Extend children to think about all possible measures e.g. length, time and the outcomes that would happen if everything was rounded to 1 decimal place e.g. everyone running 10.1 seconds in the 100 metres.

In the number 60 802.8917

- Which digit is in the thousands position?
- Which digit is in the thousandths position?

When I round 0.0020499 to 3 significant figures, the answer is

- a. 0.002
- b. 0.00205
- c. 0.00204

Explain your answer.

-3.5 is rounded to the nearest whole number.

Justin says, "3.5 rounded to the nearest whole number is 4, so -3.5 rounded to the nearest whole number is -4."

Matt says, "When the decimal is exactly half way between two integers, we round to the largest integer. -3.5 is between -3 and -4, and -3 is larger. So -3.5 to the nearest whole number is -3."

Do you agree with Matt or Justin? Explain your thinking.

NUMBER

Selected National Curriculum Programme of Study statements

Pupils should be taught to:

- Order positive and negative integers, decimals and fractions; use the number line as a model for ordering of the real numbers; use the symbols $= < > \leq \geq$
- Use integer powers and associated real roots (square, cube and higher), recognise powers of 2, 3, 4, 5 and distinguish between exact representations of roots and their decimals approximations
- Interpret and compare numbers in standard form $A \times 10^n$, $1 \leq A < 10$
- Work interchangeably with terminating decimals and their corresponding fractions (such as 3.5 and $\frac{7}{2}$ or 0.375 and $\frac{3}{8}$)
- Appreciate the infinite nature of sets of integers, real and rational numbers.

The Big Ideas

This group of statements focuses on pupils' understanding of ordering and comparing the value of numbers that are expressed in different ways.

You might like to encourage your pupils to consider the representation that makes comparison easiest in each situation, and to explore why this is their chosen representation. Some pupils may 'persist in trying to treat decimals as if they were whole numbers'⁴.

Some may manipulate the calculations and order values without seeming to need to evaluate each step of a calculation: encourage them to share and explain their insights with the rest of the class.

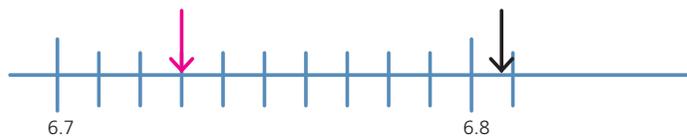
Mastery

Place these numbers on a number line:

-2.38 , 2.38 , $\frac{14}{5}$, $2\frac{3}{8}$, $-\frac{2}{5}$, $\sqrt[3]{10}$

Explain your answer.

What number do you think the red arrow is pointing to? What about the black arrow? Explain your answers.



Mastery with Greater Depth

Jo says that 187 hundredths is the same as 1870 thousandths. Explain how you know whether or not Jo is correct.

James says that $\sqrt{50} = 7.07$ and Claire says that $\sqrt{50} \approx 7$. Which is the better answer? Explain your thinking.

⁴Watson, A., Jones, K. and Pratt, D., 2014. Key Ideas in Teaching Mathematics - Reasoning with Decimals in KS3. NCETM Secondary Magazine 108. <https://www.ncetm.org.uk/resources/44075>

Put these numbers in order from smallest to largest: $1^9, 2^8, 4^5, 5^4, 7^3, 9^1$

Explain how you decided.

Complete the missing boxes using the symbols: $= < > \leq \geq$

- $5.67183 \square 0.567183 \times 10^3$
- $\frac{7}{8} \square 0.625$
- $\sqrt[4]{625} \square 5$
- $4.7 \times 10^2 \square 4.7 \times 10^4$

Explain your answers.

Which of these is correct?

- $7 \frac{187}{1000} = 7.00187$
- $77 \frac{187}{1000} = 7.187$
- $77 \frac{187}{1000} = 7.0187$

Explain your answer.

If 6 miles $\approx 10^4$ metres. Which of these is approximately 600 miles?

- a) 10^{10} metres b) 10^6 metres c) 10^8 metres

Explain your reasoning.

Fill in the box to make the calculation correct.

$$3 \times 10^4 \times \square = 6 \times 10^5$$

Explain your method.

Find a number between $-\frac{1}{9}$ and $-\frac{1}{8}$.

...and another

...and another

What generalisation can you make about the numbers between $-\frac{1}{9}$ and $-\frac{1}{8}$?

Without using a calculator, decide which two integers $\sqrt[3]{119}$ is between?

Explain your reasoning.

NUMBER

Selected National Curriculum Programme of Study statements

Pupils should be taught to:

- Use the concept of prime numbers, factors (or divisors), multiples, common factors, common multiples, highest common factor, lowest common multiple, prime factorisation, including using product notation and the unique factorisation property

The Big Ideas

This group of statements focuses on properties of number.

Marcus du Sautoy describes prime numbers as 'Nature's gift to the mathematician'⁵, and it is the understanding of prime numbers that underpins a number of the concepts being considered here.

Although many of these questions may be tackled by a simple calculation, the aim is to enable pupils to exploit the connections and relationships between the properties, and to exploit the properties to minimise the quantity and complexity of the calculations needed. Encourage pupils to share and explain their insights.

Mastery

Give reasons why none of the following are prime numbers:
4094, 1235, 5121

Find:

- a prime number greater than 100
- all of the pairs of prime numbers that add up to 98.

Explain your method.

Write down a number over 100 that is a multiple of 7.

...and that contains a 4

...and that doesn't contain a 1

Explain your method.

Mastery with Greater Depth

The highest common factor of 364 and 644 is 28.

Describe how you would simplify the fraction $\frac{364}{644}$ so that you were sure that it was in its simplest form.

The prime factorisation of 24 is $2^3 \times 3$.

The prime factorisation of 108 is $2^2 \times 3^3$.

Use this information to find the highest common factor of 24 and 108 and explain your reasoning.

What other numbers have only 2 and 3 as prime factors? Explain your thinking.

Write down a number that is one more than a multiple of seven and is also one more than a multiple of eight. Explain how you found your answer.

NUMBER

Selected National Curriculum Programme of Study statements

Pupils should be taught to:

- Use the four operations, including formal written methods, applied to integers, decimals, proper and improper fractions, and mixed numbers, both positive and negative
- Use conventional notation for the order of operations, including brackets, powers, roots and reciprocals
- Recognise and use relationship between operations including inverse operations

The Big Ideas

This group of statements focuses on arithmetic procedures.

Pupils need to develop understanding of, and hence fluency in using, operations. Acronyms, such as BIDMAS inhibit deep understanding and can cause misconceptions. Considering the order of operations, Watson (2016) writes in a blog (<https://educationblog.oup.com/secondary/maths/order-and-disorder>) about the ambiguities of BIDMAS. Watson continues to say that the 'new curriculum for primary Year 6 offers strong guidance that algebra should be introduced to express what children already know about number operations and relations; the new curriculum for Key Stage 3 includes a similar statement. If students know that ambiguities need to be sorted out, they are likely to be more willing to learn how mathematicians sort them out through precise notation. The notation follows from their mathematical needs, rather than their learning needs following from the notation.'

Mastery

A teacher asked her class to calculate $\frac{2}{3} \times \frac{1}{3}$. The answers from her students included...

- $\frac{2}{3} \times \frac{1}{3} = \frac{2}{6}$
- $\frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$
- $\frac{2}{3} \times \frac{1}{3} = \frac{3}{6}$
- $\frac{2}{3} \times \frac{1}{3} = \frac{3}{9}$

Did she get a correct answer? Explain what mistakes have might have been made for any incorrect answers.

Mastery with Greater Depth

Is it always, sometimes or never true that the use of brackets in an addition and multiplication calculation will change the value of the answer? Explain your thinking.

$$81 \times 36 = 2916.$$

Explain how you can use this fact to devise calculations with answers 29.16, 2.916, 0.2916.

Alice, Bekah and Clare are explaining why $\frac{2}{3} \div \frac{1}{3} = 2$

- Alice says "Because you turn the second number upside down and multiply, so $\frac{2}{3} \div \frac{1}{3} = \frac{2}{3} \times \frac{3}{1} = \frac{6}{3} = 2$
- Bekah says "Because if I share two thirds of a cake between one third of a person then to get a whole person I need to multiply by three, so that means that the person gets six thirds of the cake and six thirds is the same as two."
- Clare says " $\frac{2}{3} \div \frac{1}{3}$ means 'how many one thirds are there in two thirds?' Because two thirds is the same as $2 \times \frac{1}{3}$, the answer must be 2"

Which explanation do you find most convincing? Why?

Is it true or false that $-5^2 = -(5^2) = (-5)^2$?

Explain your thinking.

Calculate:

- $3.2 \times (8.6 + 4.4)$
- $3.2 \times 8.6 + 4.4$
- $3.2 + 8.6 \times 4.4$
- $(3.2 + 8.6) \times 4.4$

Explain your methods.

$$81 \times 36 = 2916.$$

Explain how you can use this fact to solve:

$$8.1 \times 36$$

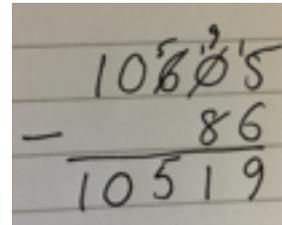
$$0.81 \times 360$$

$$29.16 \div 36$$

$$81 \times 3.7$$

and solve each of the calculations.

Joe solved the following calculation:


$$\begin{array}{r} 10519 \\ - \quad 86 \\ \hline 10519 \end{array}$$

Claire said that she had a quicker way to do this calculation.

What do you think she means?

Will the answer to $62 \div 0.9$ be smaller or larger than 62?

Explain how you know.

Is the following statement always, sometimes or never true? Explain your reasoning.

'Addition and multiplication make numbers bigger, subtraction and division make numbers smaller.'

Explain why someone might work out $10 - 2 + 5$ as 3. What mistake have they made?

$$220.5 \div 4.9 = 45$$

Explain how you can use this fact to solve:

$$490 \times 4.5$$

$$2205 \div 49$$

$$4.4 \times 0.49$$

and solve each of the calculations.

Before Hannah was paid, her bank balance was -£104.38; after she was paid, her balance was £1312.86. How much was she paid? Show your calculation on a number line.

Explain why someone might work out $3 + 2 \times 5$ as 25. What mistake have they made?

NUMBER

Selected National Curriculum Programme of Study statements

Pupils should be taught to:

- Use a calculator and other technologies to calculate results accurately and then interpret them appropriately

The Big Ideas

This group of statements focuses on the use of a calculator.

Analysis of exams often suggests an overreliance on, and a lack of care when using calculators⁶. The questions here seek to expose the ways in which your pupils use their calculators. Look particularly for premature rounding or approximation when calculating, ensuring that pupils are able to use their calculators efficiently.

Mastery

Hannah wants to work out the square of -5.



Explain what she has done wrong.

Use a calculator to work out the following:

$$\frac{9}{13} + \frac{6}{19}$$

$$\frac{3}{8} \text{ of } \pounds 16.27$$

Explain your method and how you know whether your answer is likely to be right.

Mastery with Greater Depth

Which is the odd one out, and why:

- $\sqrt[3]{59.319}$
- 40% of 19.5
- $18.72 \div \sqrt{23.04}$

Change one of the calculations to make them all equal.⁷

Willow is working out the calculation $(\sqrt{2} + 3.8)^2$.

She uses her calculator to find $\sqrt{2}$



and then uses this to calculate a final answer



Comment on Willow's answer.

⁶For example, Edexcel Chief Examiner's report for trials of the new GCSE (2016) and OCR Chief Examiner's report of A-level maths (2015)

⁷Look out for the sophistication of the changes that pupils make and the depth of understanding that may underpin this. For example, changing 40% to 20% rather than completely changing the question.

ALGEBRA

Selected National Curriculum Programme of Study statements

Pupils should be taught to:

- use and interpret algebraic notation, including:
 - ab in place of $a \times b$
 - $3y$ in place of $y + y + y$ and $3 \times y$
 - a^2 in place of $a \times a$; a^3 in place of $a \times a \times a$; a^2b in place of $a \times a \times b$
 - $\frac{a}{b}$ in place of $a \div b$
 - coefficients written as fractions rather than as decimals
 - brackets
- understand and use the concepts and vocabulary of expressions, equations, inequalities, terms and factors

The Big Ideas

This group of statement focuses on terminology and notation.

The key idea here is that pupils understand that letters represent numbers and pupils can give a reason why, for example, $2n$ and n^2 are different by explaining the values these expressions take for different values of n .

By working on the 'grammar' of the symbols, by constructing symbolic representations themselves and by translating between words and symbols pupils come to appreciate what the symbols are symbolising. Jones, Pratt & Watson (2013)⁸ note that 'students' problems with early algebra mainly stem from incorrect interpretation of notation. As with any notation, it is more effective to know what you are expressing before having to use standard notation to do so.'

Pupils need to appreciate the difference between an equation, which is true for some values of the variable (i.e. sometimes) and an identity, which is true all of the time because it is just another way of writing the expression.

It is also important for pupils to be able to see expressions like $2x + 5$ as both an instruction to operate on an unknown quantity x (i.e. double and add 5) and the result of performing that operation so that $2x + 5$ can be regarded as a single mathematical object. However, pupils' experiences up to this point may reinforce the view that they should carry out an operation in order to reach a solution (leading to pupils reducing $2x + 5$ to $7x$ for example). The aim is to enable pupils to demonstrate flexibility in thinking about an expression as both a process and as a conceptual object (This has been labelled as 'proceptual' thinking by Grey and Tall, (1994))⁹.

⁸Watson, A., Jones, K. and Pratt, D., 2014. Key Ideas in Teaching Mathematics - Algebraic Reasoning. NCETM Secondary Magazine 109. <https://www.ncetm.org.uk/resources/44331>

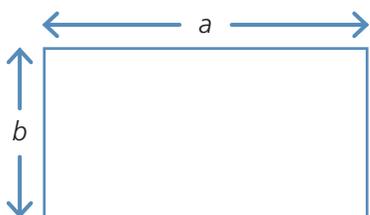
⁹Gray, E.M. and Tall, D.O., 1994. Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic. Journal for research in Mathematics Education, pp.116-140.

Mastery

Which is bigger: $2n$ or n^2 ? Explain your answer.

When is a^2b bigger than ab^2 ? Explain your answer.

Write expressions for the perimeter of this rectangle in different ways.



Are these statements, always, sometimes or never true?

$$2x + 3 = 2x + 6$$

$$2x + 6 = 2(x + 3)$$

$$2x + 6 = 3x + 2$$

$$2x + 3 > 2x + 6$$

$$3x + 2 < 2x + 6$$

Explain why you have decided on each answer and for those that are "sometimes true", explain when they are true.

Mastery with Greater Depth

Match up expressions on the left with their corresponding description in words on the right.

$4 + 2x$ Four less than x

$x - 4$ Four times the number that is two more than x

$2x - 4$ Two less than one quarter of x

$(x + 2) \div 4$ Four more than twice x

$4(x + 2)$ One quarter of the number which is two more than x

One expression on the left and one description on the right can't be matched.

Can you write a description for the expression that isn't matched?

Can you write an expression for the description that isn't matched?

$$2x + 6 = 2(x + 3)$$

$$2x + 6 = 3x + 2$$

What is the same and what is different?

ALGEBRA

Selected National Curriculum Programme of Study statements

Pupils should be taught to:

- simplify and manipulate algebraic expressions to maintain equivalence by: collecting like terms; multiplying a single term over a bracket; taking out common factors; expanding products of two or more binomials
- understand and use standard mathematical formulae; rearrange formulae to change the subject

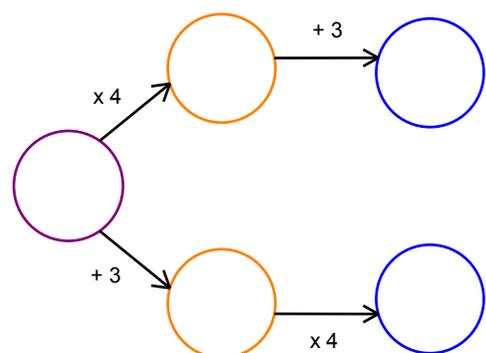
The Big Ideas

These statements focus on simplifying and manipulating expressions, equations and formulae.

Pupils need to understand the meaning of certain algebraic manipulations as well as how to perform the manipulation e.g. what multiplying across a bracket actually means; why, when you subtract $(n - 3)$ it is the same as subtracting n and adding 3.

Pupils need to understand that a formula is another way of representing the relationship between numerical quantities and that the reasoning behind changing the subject of a formula is the same as that behind solving an equation: i.e. making l the subject of the formula $P = 2(l + w)$ is similar to solving the equation $2(x + 3) = 10$.

Mastery



Choose a number to go in the purple circle (extreme left) and fill in the other circles accordingly.

Explain why the difference between each pair of answers in the blue circles (extreme right) is always 9.

Write an algebraic identity to show why this is true.

Mastery with Greater Depth

Which of these statements are true and which are false?

$$4(b + 2) = 4b + 2$$

$$3(p - 4) = 3p - 7$$

$$2(x + 7) + 6 = 2x + 20$$

$$-2(5 - b) = -10 - 2b$$

$$12 - (n - 3) = 15 - n$$

For those that are true, give a reason and/or draw a picture¹⁰ for why they are true.

For those that are false, explain why they are false and correct them.

¹⁰Some pupils may not be familiar with using images to represent algebraic relationships. You might like to prompt them by suggesting possibilities such as bar models, number lines, an array/grid image. You may find some of this session [<https://content.ncetm.org.uk/itt/sec/KeelePGCEMaths2006/StandardsUnit/ImprovingLearning/A1.pdf>] from "Improving learning in mathematics" offers some ideas on supporting students to use other representations.

Make l or w the subject of the formula: $P = 2(l + w)$
Explain each step.

Make C the subject of the formula: $F = \frac{9C}{5} + 32$
Explain each step.

Make r the subject of the formula: $A = \pi r^2$
Explain each step.

The formula $F = \frac{9C}{5} + 32$ describes how to convert degrees Centigrade into degrees Fahrenheit.

When you use this formula with a certain temperature in $^{\circ}\text{C}$, you get the same value in $^{\circ}\text{F}$. What is this value? Explain your thinking and your method.

Complete the blanks in three different ways:

$$_ + 6x + _ = (_ + _) (_ + _)$$

$$_ + 6x + _ = (_ + _) (_ + _)$$

$$_ + 6x + _ = (_ + _) (_ + _)$$

(Adapted from: Arcavi, A., Drijvers, P. and Stacey, K., 2016. The Learning and Teaching of Algebra: Ideas, Insights and Activities. Routledge.)

ALGEBRA

Selected National Curriculum Programme of Study statements

Pupils should be taught to:

- work with coordinates in all four quadrants
- recognise, sketch and produce graphs of linear and quadratic functions of one variable with appropriate scaling, using equations in x and y and the Cartesian plane
- interpret mathematical relationships both algebraically and graphically
- reduce a given linear equation in two variables to the standard form $y = mx + c$; calculate and interpret gradients and intercepts of graphs of such linear equations numerically, graphically and algebraically
- use linear and quadratic graphs to estimate values of y for given values of x and vice versa and to find approximate solutions of simultaneous linear equations

The Big Ideas

These statements focus on graphical representation.

The KS3 programme of study states that pupils should be taught to 'move freely between different numerical, algebraic, graphical and diagrammatic representations' and to 'express relationships between variables algebraically and graphically.'

It is important for pupils to have opportunities to reason with coordinates and the coordinate system and not just practise plotting coordinates. Through such reasoning, pupils will come to gain a deeper understanding of the relationship between the x and y coordinates of all points on a line and the equation of that line.

Mastery

The opposite vertices of a rectangle are plotted at $(0,2)$ and $(6,3)$.
Find the coordinates of the other two vertices.

Which of the following points lie on the line $2x + y = 7$:
 $(1,5)$ $(5,1)$ $(3,1)$ $(4,1)$ $(5,3)$ $(5,-3)$?

Can you explain why or why not?

Can you show this with a calculation as well as a drawing?

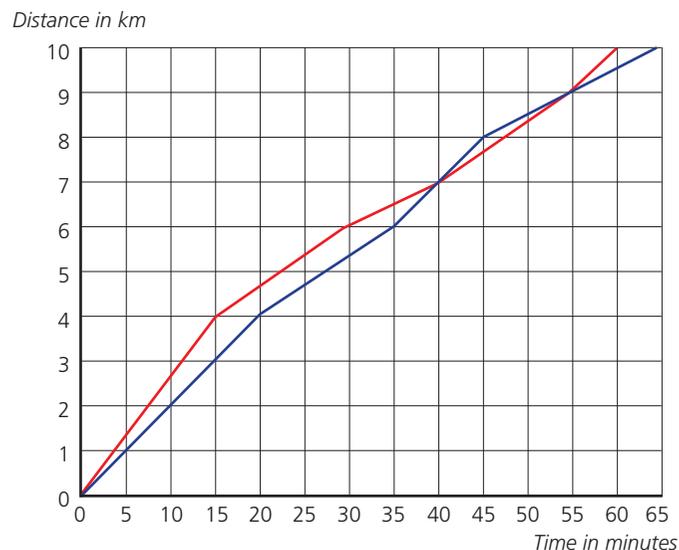
Mastery with Greater Depth

Two vertices of a square are plotted at the points $(2,3)$ and $(6,3)$.
Find the other two coordinates.
In how many ways can this be done?

What linear equations might produce the following patterns of straight lines?



Two women, Rose and Violet, are running a 10km road race. Rose is shown in red and Violet is shown in blue.



- Who starts off faster? How do you know?
- Do they ever run at the same speed? How do you know?
- Did one ever overtake the other? When?
- Who wins the race?
- How far behind her was the loser?

Draw a line which is parallel to the line $2x + y = 8$ and write down the equation of that line;

... and another,
and another,
and another, ...

How do you know they are parallel?

What quadratic equations might produce the following types of curves?



Draw and write down the equations of three lines which, when drawn with the line $y = 2x + 1$ produce a tilted 'noughts and crosses' board similar to this:



What helps you to decide whether to use an algebraic or a graphical method to solve a pair of simultaneous equations?

Is it possible for a pair of simultaneous equations to have two different pairs of solutions or to have no solution? How do you know?

How does a graphical representation help you to know more about the number of solutions?

Which of these equations represent the same straight line:

$$2x + y = 8$$

$$y = 2x + 8$$

$$y + 8 = 2x$$

$$y = 2x - 8$$

$$x = \frac{1}{2}y - 8$$

$$y = -2x + 8$$

Explain your answers using words and calculations as well as graphs.

Given that x and y satisfy the equation $5x + y = 49$ and $y = 2x$, find the value of x and y using an algebraic method.

Solve graphically the simultaneous equations

$$x + 3y = 11 \text{ and } 5x - 2y = 4$$

ALGEBRA

Selected National Curriculum Programme of Study statements

Pupils should be taught to:

- generate terms of a sequence from either a term-to-term or a position-to-term rule
- recognise arithmetic sequences and find the n th term
- recognise geometric sequences and appreciate other sequences that arise

The Big Ideas

These statements focus on sequences.

Sequences offer pupils opportunities to explore and describe mathematical structure, not just look for the patterns that the structure creates. It is important that this underpinning structure, the sequence it generates and the way in which the structure is represented in algebraic notation are all explored by the pupils. When drawing or building sequences of matchsticks or counters, and finding the n th term, attention should be given to identifying the connection between the notation and the image. For example, if a matchstick pattern represents $3n + 1$, where is the '3' in the pattern of matchsticks? Where is the '1'?

Mastery

A sequence is generated using the rule $u_n = 4n - 1$ (i.e. the n th term is $4n - 1$).
Is the number 136 in this sequence? Explain how you know.

Which of these are arithmetic sequences:

1, 3, 5, 7, 9, ...

12, 15, 18, 21, ...

1, 1, 2, 3, 5, 8, ...

47, 37, 27, 17, 7, ...

5, 1, 5, 1, 5, 1, 5, ...

2, 4, 8, 16, ...

... and why?

What are their n th terms?

Mastery with Greater Depth

The term-to-term rule for a sequence is $u_{n+1} = 2u_n + 3$ with $u_1 = 7$.

Generate some terms for this sequence and explain any patterns that you see.

Write down an arithmetic sequence that has the numbers 9 and 25 in it (but not as consecutive numbers)

... and another,

... and another, ...

What is the n th term of each of these sequences?

Which of these are geometric sequences:

1, 2, 4, 8, 16, ...

200, 20, 2, 0.2, ...

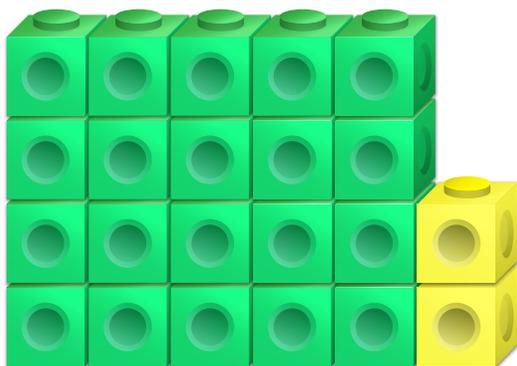
5, 15, 15, 45, 50, 150, ...

8, 12, 18, 27, ...

... and why?

Continue each one for 4 more terms.

This picture shows the 5th term of a pattern made with cubes to represent the sequence $4n + 2$.



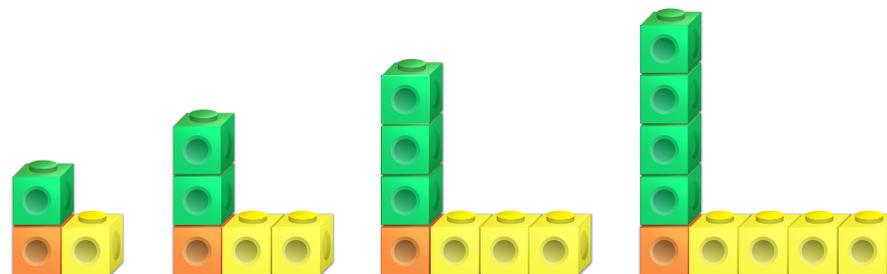
- What in the picture shows that it's the 5th term?
- What in the picture shows that $4n$ is a part of the rule for the sequence?
- What in the picture shows that $+2$ is a part of the rule for the sequence?

Find a geometric sequence that begins with a number between 1 and 10 and reaches or exceeds 1000 after exactly 7 terms.

... and another,

... and another,

This picture shows the first four patterns in a growing pattern of L shapes.



Arjan says, "The number of cubes in the n th term will be $2n + 1$ "

Brett says, "The number of cubes in the n th term will be $(n+1) + n$ "

Ceri says, "The number of cubes in the n th term will be $(n+1)^2 - n^2$ "

Explain how each of them is interpreting the diagram and show that all of them are correct.

ALGEBRA

Selected National Curriculum Programme of Study statements

Pupils should be taught to:

- use algebraic methods to solve linear equations in one variable (including all forms that require rearrangement)
- substitute numerical values into formulae and expressions, including scientific formulae

The Big Ideas

These statements focus on solving equations.

Understanding both the = sign and the correct use of order of operations, along with inverse operations are keys to the solving of equations. Pupils also need to understand the impact of the = sign as they move from expressions to equations. This can be problematic, as pupils may have built an understanding that a variable can represent 'any number' and are now being asked to find just one correct value.

Pupils should experience doing and undoing in the context of equations (i.e. 'building up' equations by starting with a simple " $x = 3$ ", say and developing this by operating on both sides to create more and more complex equations) as well as unpicking them to reach a solution.

Mastery

Solve:

$$3c - 7 = -13$$

$$4(z + 5) = 84$$

$$4(b - 1) - 5(b + 1) = 0$$

$$\frac{21}{x+4} = \frac{21}{x+4}$$

Explain each step.

Mastery with Greater Depth

How do you decide where to start when solving a linear equation?

Given the list of linear equations in the first column decide:

- Which of these are easy to solve?
- Which are difficult and why?

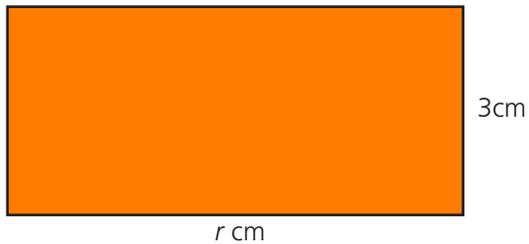
What strategies are important with the difficult ones?

When you substitute $a = 2$ and $b = 7$ into the formula $t = ab + 2a$ you get 18.

Can you make up some more formulae that also give $t = 18$ when $a = 2$ and $b = 7$ are substituted?

Explain how the equation $2r + 6 = 22$ might relate to the diagram below.

- What is the perimeter of the rectangle? How do you know?
- How wide is the rectangle?



Pete is solving a linear equation. He draws this bar model to help.

- What equation is Pete solving?
- What is the value of t ?

t	t	t	7
t	t	10	

Explain how you know.

ALGEBRA

Selected National Curriculum Programme of Study statements

Pupils should be taught to:

- model situations or procedures by translating them into algebraic expressions or formulae and by using graphs

The Big Ideas

These statements focus on using algebra to model situations.

The situations here offer a structure to give meaning to the algebraic notation. It is important that pupils understand that the same relationship can be written correctly in different ways (for example $2(m+n)$, $m+m+n+n$, $2m + 2n...$) and you may wish to ask pupils to describe the different ways of 'seeing' that the different algebraic representations describe.

Mastery

There were m boys and n girls in a parade. Each person carried 2 balloons. Which of these expressions represents the total number of balloons carried in the parade?

- A $2(m + n)$
- B $2 + (m + n)$
- C $2m + n$
- D $2n + 2m$
- E $m + 2n$

Pick at least one answer that doesn't represent this statement and describe the scenario that it does describe.

Mastery with Greater Depth

A piece of wood was 40cm long. It was cut into 3 pieces. The lengths in cm are: $2x - 5$, $x + 7$, $x + 6$.

What is the length of the longest piece? Explain your answer.

Plums are £3 a bag and melons are £2 each.

If b stands for the number of bags of plums bought and m stands for the number of melons bought, what does $3b + 2m$ stand for?

Each bag of plums contains 6 plums. What is the total number of pieces of fruit bought? Explain your answer.

RATIO, PROPORTION AND RATES OF CHANGE

Selected National Curriculum Programme of Study statements

Pupils should be taught to:

- Define percentage as 'number of parts per hundred', interpret percentages and percentage changes as a fraction or a decimal, interpret these multiplicatively, express one quantity as a percentage of another, compare two quantities using percentages, and work with percentages greater than 100%
- Solve problems involving percentage change, including: percentage increase, decrease and original value problems and simple interest in financial mathematics

The Big Ideas

These statements focus on percentages and include percentage changes.

Although the use of informal methods (such as calculating 10% or 1% and combining them to find a percentage of a quantity) will successfully give a solution to a question it is not always the most efficient method, nor does it support pupils' understanding of percentage change. As identified by Watson, Jones and Pratt (2013)¹¹, "If learners only think of multiplication as to do with repeated addition, arrays, and 'times tables' it is hard to apply multiplication to proportional relationships."

Encourage pupils to use the more efficient method of using a multiplier to work with percentages and percentage changes.

Mastery

Convert the following percentages into fully simplified fractions:

85%, 192%, 3.75%

Five people tried to calculate 18.3% of 541. Here is what they typed into their calculators:

a) 541×1.83 b) 0.183×541

c) 541×0.183 d) $541 \div 0.183$

e) $541 \div 100 \times 18.3$ f) 541×18.3

Which of these will give me the correct answer? For the incorrect answers, can you write a percentage question that their calculation does give the answer to?

Mastery with Greater Depth

What is 20% of 30% of 50% of £320?

Can you find one percentage that is equivalent to this?

Can you explain why?

Which is greater 120% of 90 or 90% of 120? Comment on your results.

Freddie ate one-fifth of a pizza. He then split what was left into five equal pieces for his friends. William ate two of those pieces and then claimed that 60% of the pizza was left. Is he right? Explain your reasoning.

¹¹Watson, A., Jones, K. and Pratt, D., 2014. Key Ideas in Teaching Mathematics - Multiplicative Relationships. <http://www.nuffieldfoundation.org/key-ideas-teaching-mathematics/multiplicative-relationships>

A charity needs to raise £4600 for their latest project. They currently have £2990. What percentage of the total is left to raise?

Wade's test scores are given below. In which test did Wade score the highest percentage?

Test 1: 82%

Test 2: 16 out of 20

Explain how you know.

A special edition of a chocolate bar is 20% bigger than the normal bar for an advertising campaign. After the campaign the size of the special edition bar is decreased by 20%.

Is the chocolate bar now bigger, smaller or the same as before the special edition was produced? Explain how you decide.



How long is a normal issue of the magazine pictured to the left?

Zlatan is given £500 by his grandma for his 14th birthday. He wants to save it, so that he can afford driving lessons on his 17th birthday. Which option should he go for?

- Holmes Chapel Bank: 20% bonus immediately added, then 5% simple interest per year.
- Holmes Chapel Building Society: 7% annual simple interest, then a 15% bonus when you withdraw.

Explain how you decided.

Is a 50% increase followed by a 50% increase the same as doubling? Explain your answer.

James and Lily want to book flights to Portugal. When they last checked the flights cost £320. Now they cost £640. James claims that this is a 200% increase in price, whereas Lily thinks it's a 100% increase. Who do you think is right and why?

Reed: "I ate 110% of the pizza."

Johnny: "My foot has swollen to 110% of its normal size."

Ben: "That cake I ate contained 110% of my daily sugar allowance."

Who is definitely lying and why?

x is 50% of y .

y is 90% of z .

What percentage of z is x ? Explain your thinking.

Richard wants to write a formula that tells you how to calculate the percentage increase for any amount and any percentage. He begins by writing, let x = original amount and let y = percentage increase. Can you finish it off?

RATIO, PROPORTION AND RATES OF CHANGE

Selected National Curriculum Programme of Study statements

Pupils should be taught to:

- Change freely between related standard units [for example time, length, area, volume/capacity, mass]
- Use compound units such as speed, unit pricing and density to solve problems
- Use and interpret scale factors, scale diagrams and maps

The Big Ideas

These statements focus on units of measurement.

It is important to note that measure is a critical context through which pupils are able to explore and show their understanding of direct proportion and ratio. Watson, Jones and Pratt (2013)¹² describe measures as “numbers that are connected to a quantity, the connection being a ratio comparison (and not simply a count of a number of ‘things’)”. Pupils need to explore these ratios and the multiplicative relationships that underpin them, understanding that converting between units for time, length, mass etc, or calculations involving speed, density, or unit pricing are all examples of direct proportion.

Mastery

Peter and Miles are racing to complete a video game. It took Peter 6 hours and 15 minutes. Miles took 6.2 hours. Peter says ‘I win, as I was 5 minutes quicker than you.’ Comment on Peter’s judgement.

A square tile has an area of 1cm^2 . I want to make a mosaic picture that is $1\text{m} \times 1\text{m}$. Tiles cost 13p each. How much would the mosaic cost to make? Explain your answer.

Mastery with Greater Depth

A rectangular room measures 6 metres by 4 metres. I want a scale drawing, with $1\text{cm} = 0.25\text{m}$. What is the area of the room? What is the area of my scale drawing? Explain your answer.

¹²Watson, A., Jones, K. and Pratt, D., 2014. Key Ideas in Teaching Mathematics - Connecting Measurement and Decimals. <http://www.nuffieldfoundation.org/key-ideas-teaching-mathematics/connecting-measurement-and-decimals>

A millipede went 0.63km in 2 hours.
In 20 minutes a centipede travelled 106m.
It took an hour and a half for an earwig to do 47 400cm.
Which was quickest? Explain your answer.

The density of water is 0.997g/cm^3 . If an object is less dense than water it floats. Will the following sink or float?

- a) An 8000mm^3 lump of carbon weighing 28g.
- b) A 0.16kg orange with a volume of 156cm^3 .

If a banana has a mass of 150g and floats, what can you say about its volume?

A rectangular room measures 6 metres by 4 metres. I want a scale drawing, with $1\text{cm} = 0.25\text{m}$. What will the dimensions of my scale drawing be?

Describe an object that is designed to increase pressure.
Describe an object that is used to decrease pressure.

Which has more detail: a map with a scale of 1:10 or a map with a scale of 1:100? Why?’

A greyhound can run at a top speed of 19m/s. The top speed of a wild ass is 64km/h. A horse can run at 55mph. (5 miles = 8km).

Order these animals in ascending order of speed. Explain how you reached your conclusion.

Given that 5 miles = 8km, sketch a conversion graph showing how to convert distances between miles and km. Mark on as many points as you can so that conversions can be carried out up to 20 miles or 32km.

RATIO, PROPORTION AND RATES OF CHANGE

Selected National Curriculum Programme of Study statements

Pupils should be taught to:

- Use ratio notation, including reduction to simplest form
- Divide a given quantity into two parts in a given part:part or part:whole ratio; express the division of a quantity into two parts as a ratio

The Big Ideas

These statements focus on ratio.

As before, the aim is to enable pupils to use the multiplicative relationships within ratios rather than using additive methods.

Some pupils may be fluent with the use of the bar model, or other robust visual representation, to tackle problems involving sharing quantities in a given ratio, For others such an image may support them in reaching a solution, as well as those who no longer need such scaffolding.

Mastery

There were fifty-six men and sixteen women on a bus, so the ratio of men to women is 7:2.

At the last stop ten women and three men got on the bus. Eight men and two women got off. What was the ratio of women to men after this stop?

Explain your answer.

Fill in the boxes.

$$24:16 = \square:\square = 3:\square$$

$$1.5:30 = 1:\square$$

$$3.2\text{kg}:415\text{g} = \square:415 = 640:\square$$

Which was the hardest box to fill in? Explain why.

Mastery with Greater Depth

Grace is counting up in twos and Max is counting up in fives.

Grace says "Two" and Max says "Five".

Grace says "Four" and Max says "Ten".

After a while, Max says "Eighty five." What has Grace just said?

Explain how you know.

If the ratio of boys to girls in a class is 7:2, could there be exactly 27 children in the class? Why? Could there be 25 boys? Why?

Natasha and Clint share £44 in the ratio 3:5. How much more money does Clint get than Natasha? Use a bar model to represent the situation.

Purple paint is made using 1 litre of blue paint for every three litres of red. What is the ratio of red:blue paint? Use a bar model to represent the situation.

Craig makes orange squash using 7 parts of water for every 3 parts of squash. Jess makes orange squash using 6 parts of water for every 2 parts of squash. Does Craig or Jess make the most orangey squash? Explain how you know.

On a times table square, which numbers are directly above and directly below 56 in the 7s row? Explain how you know.

Frank and Matt are playing a card game and start by sharing cards between them. The total number of cards in play does not change through the game. Halfway through, the ratio of the cards they each hold is 5:9. When they stop the game the ratio is 8:13. Did Frank win or lose cards in the second half of the game?

The sum and difference of two integers are in the ratio 5:1 respectively. How many possible integer pairs can you find? Can you spot a pattern to the answers?

RATIO, PROPORTION AND RATES OF CHANGE

Selected National Curriculum Programme of Study statements

Pupils should be taught to:

- Solve problems involving direct and inverse proportion, including graphical and algebraic representations

The Big Ideas

These statements focus on proportion.

As Mason (2008)¹³ identifies, multiplicative reasoning 'depends on multiplication seen as a scaling rather than simply as repeated addition' and proportion offers a useful context for this. While working on these questions, you might look for those pupils who have mastered the use of multiplicative approaches, while focusing on those who are using a repeated addition image for multiplication.

It is also critical for pupils to see proportion in a wide variety of contexts (see the previous section on units of measure) and in different representations. The aim is to enable pupils to move fluently between different representations (such as graphs, tables of values and algebraic representations) and to explain different representations for proportional relationships.

Mastery

Recipe for 8 Yorkshire Puddings:

2 cups of flour

4 eggs

200ml milk

How many eggs would be needed for 56 Yorkshire puddings?

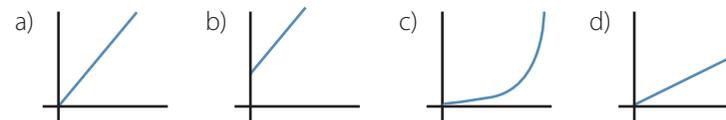
How much milk would be needed for 6 Yorkshire puddings?

To find the number of cups of flour for 216 Yorkshire puddings, Raj divided 216 by 8 and then multiplied the answer by 2. Jo divided 2 by 8 and multiplied the answer by 216. Are they both correct? Try to explain their thinking.

Mastery with Greater Depth

The n^{th} term of a sequence is $5n + 1$. The 10th term of the sequence is 51. Albert says, "In that case, the 100th term must be 510". Is Albert right? Explain your reasoning.

Which of these graphs could represent direct proportion? How do you know?



¹³Mason, J, 2008. Imagery and Awareness. Proceedings of the British Society for Research into Learning Mathematics, 28(2), 61-66.
<http://www.bsrlm.org.uk/wp-content/uploads/2016/02/BSRLM-IP-28-2-11.pdf>

Three maggots eat an apple in two days. How much longer would it have taken if there were only one maggot? Explain your answer

Complete the sentence:

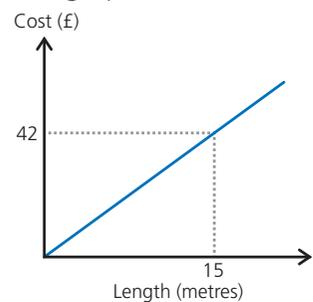
A rectangle had an area of 42cm^2 . The base was tripled and the height was _____. The area is still 42cm^2 . Explain your answer.

There are 6 bread rolls in a pack. Which equation correctly links the number of rolls (r) with the number of packs (p)?

- a) $p = 6 + r$
- b) $6r = p$
- c) $r = 6p$
- d) $p \div 6 = r$

How do you know?

The graph below shows the cost of ribbon.



How much would 60m of ribbon cost?

How much could be bought with £420?

How many whole metres of ribbon could be bought with £100?

Explain how you know.

In the numbers in Set 1, 2 and 3 are in direct proportion. Complete the missing entries.

Set 1	Set 2	Set 3
8		12
	18	
5	20	
	88	33
6		

Could you complete the grid if any of these numbers were deleted? Which ones and why?

Which of these tables of values are in direct proportion?

A	1	2	3	4
B	5	10	15	20

G	10	13	42	7
H	32	41.6	134.4	22.4

P	5	10	15	20
Q	6	11	16	21

X	8	9	10	11
Y	5	5.625	6.25	6.875

How many different ways can you show that the table(s) of values are in proportion?

Which of these formulae represent a directly proportional relationship?

- $y = 3x$
- $y = x + 3$
- $y = 3 - x$
- $y = x \div 3$
- $y = 3(x + 3)$

Explain your answer.

Three builders can complete an extension in 10 days. How long would it take six builders? One builder? Thirty builders? How realistic are these answers?

GEOMETRY AND MEASURES

Selected National Curriculum Programme of Study statements

Pupils should be taught to:

- Derive and apply formulae to calculate and solve problems involving: perimeter and area of triangles, parallelograms, trapezia, volume of cuboids (including cubes) and other prisms (including cylinders)
- Calculate and solve problems involving: perimeters of 2-D shapes (including circles), areas of circles and composite shapes

The Big Ideas

These statements focus on area and perimeter.

Skemp (1976)¹⁴ uses area as an example to demonstrate the difference between relational and instrumental understanding. The use of questions that involve reasoning about area and perimeter will allow pupils to demonstrate their understanding and fluency in both of these areas.

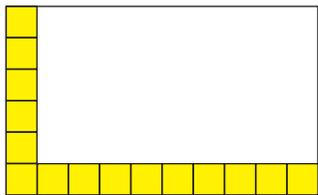
Mastery

A rectangle has an area of 6cm^2 .

Its height is not a whole number, but the length of its base is.

What might be the dimensions of the rectangle? How many possible solutions can you find? Explain your reasoning.

This rectangle is being covered with square tiles.



How many more of the square tiles are needed to cover the entire rectangle? Explain your answer.

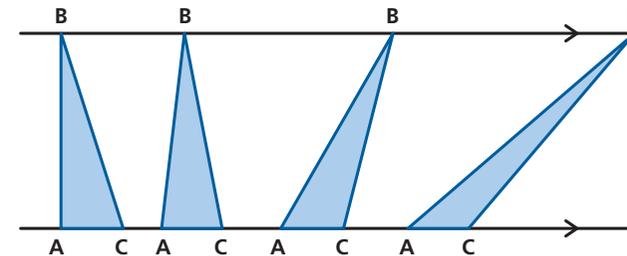
Mastery with Greater Depth

A rectangle has an area of 6cm^2 .

Its height is not a whole number, but the length of its base is.

What is the shortest perimeter that the rectangle can have? Explain how you know.

A piece of elastic is fixed at two points A and C and point B slides along the line.



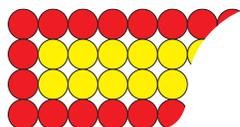
Some of the triangles that can be made are shown here.

Which triangle has the greatest area? How do you know?

¹⁴Skemp, R.R., 1976. Relational understanding and instrumental understanding. *Mathematics Teaching*, 77(1), pp.20-26.

Draw a triangle/parallelogram with an area of 6cm^2 .
 Draw a different triangle/parallelogram with an area of 6cm^2 .
 And another...

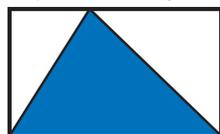
This is a section of a rectangle of yellow counters surrounded by red counters. The red rectangle is m counters long.



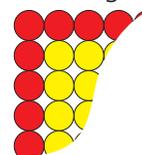
Craig says "The formula to find the total number of red counters is $2m + 4$ "
 Jess says "The formula to find the total number of red counters is $8 + 2(m - 2)$ "
 Pete says "You're both right".
 Who do you agree with?
 Explain how you know.

Sarah says, "To find the volume of a shape, you multiply all of the lengths together."
 Do you agree with Sarah?
 Explain your decision.

What fraction of the rectangle below is shaded blue?
 Explain how you know.

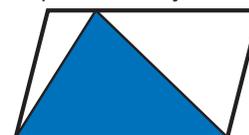


Some yellow counters are arranged in a square, then a layer of red counters is arranged to surround them, like this...



Are there always more yellow counters than red counters in the pattern?
 Or more red than yellow?
 Explain your answer.

What fraction of the parallelogram is shaded blue?
 Explain how you know.



Matilda and Louis both want pepperoni pizza. Louis says, "My favourite part is the stuffed crust." They look at this advert:

Stuffed Crust Pizzas
 Small (8 inch): £8
 Large (12 inch): £12

Special offer! Buy two small stuffed crust pizzas for £12

What should they order? Why?

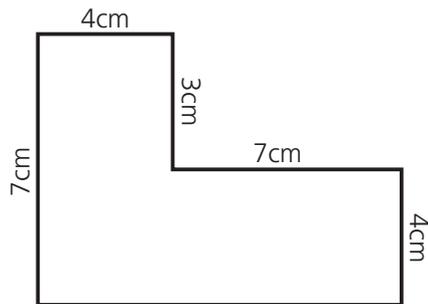
Elinor and Benedict both want stuffed crust pepperoni pizza. Should they order one 12 inch to share, or have an 8 inch pizza each? Why?"

Stuffed Crust Pizzas

Small (8 inch): £8
Large (12 inch): £12



The shape below is not drawn to scale.

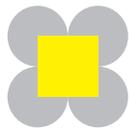


Find at least two ways to calculate the total area of the L-shape.

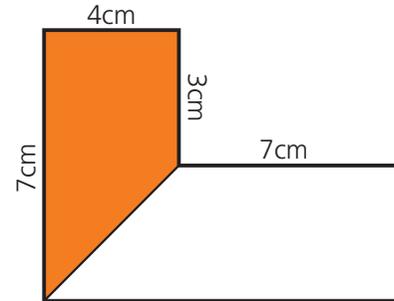
The yellow area is 16cm^2 .

- What's the perimeter of the yellow section?
- What's the perimeter of the grey section?

Explain your reasoning.



The shape below is not drawn to scale.



One way to calculate the total area of the L-shape is

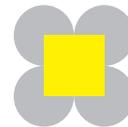
$$\frac{4 \times (3 + 7)}{2} + \frac{4 \times (7 + 11)}{2}$$

Explain why this method works.

Find another (maybe more efficient) method to calculate the total area of the shape.

Using π as 3.14 gives the grey area as 150.72cm^2

What's the perimeter of the yellow section?



GEOMETRY AND MEASURES

Selected National Curriculum Programme of Study statements

Pupils should be taught to:

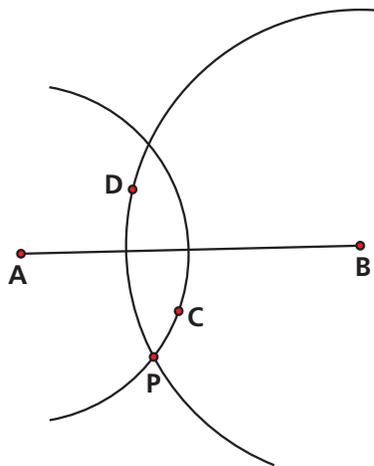
- Derive and use the standard ruler and compass constructions (perpendicular bisector of a line segment, constructing a perpendicular to a given line from/at a given point, bisecting a given angle); recognise and use the perpendicular distance from a point to a line as the shortest distance to the line
- Describe, sketch and draw using conventional terms and notations: points, lines, parallel lines, perpendicular lines, right angles, regular polygons, and other polygons that are reflectively and rotationally symmetric
- Derive and illustrate properties of triangles, quadrilaterals, circles, and other plane figures [for example, equal lengths and angles] using appropriate language and technologies

The Big Ideas

These statements focus on geometrical properties.

The KS3 Programme of study states that pupils should 'begin to reason deductively in geometry, number and algebra, including using geometrical constructions'. The aim is to justify why a given construction works as well as describe the steps needed to make that construction. Pupils' descriptions of the structures and rules that underpin the constructions offer a useful insight into their understanding of locus and the properties of simple geometric shapes.

Mastery



In this diagram:
Point C is 4cm from A and 5cm from B.
Point D is 3cm from A and 6cm from B.
How far is point P from A and from B?
Explain how you decided.

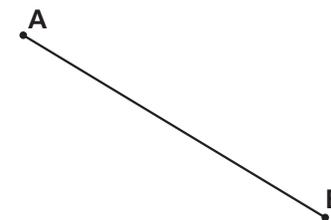
Mastery with Greater Depth

When constructing a perpendicular bisector, explain why it is important that the compasses are set the same distance apart for the whole construction.

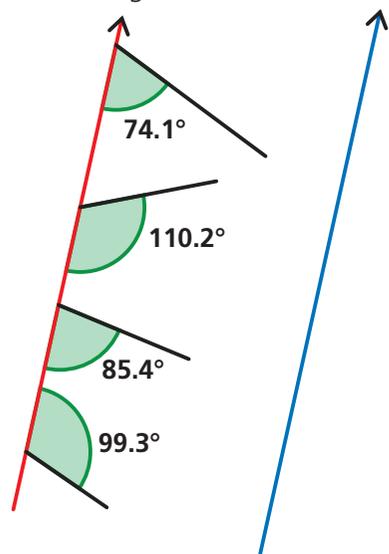
The line AB below joins two parallel lines.

It is the shortest possible line that will join these lines.

Draw the parallel lines on the diagram as accurately as you can and explain why you chose those.



In the diagram below, the red and the blue lines are parallel.



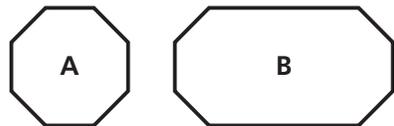
Some straight lines are being drawn from the red line to the blue line. Which of the lines will be the shortest when it reaches the blue line?

Write a set of instructions explaining how to bisect an angle.

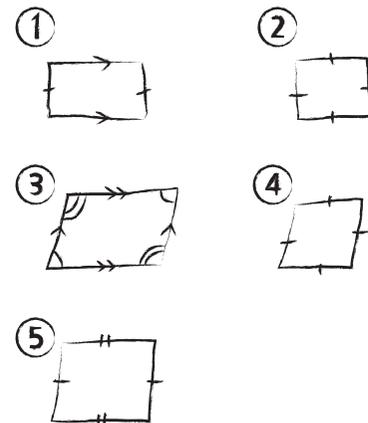
Shapes A and B are octagons. Shape A is regular.

Sketch A and B and use correct notation to mark the sides of equal length, the equal angles and any edges that are parallel in each shape.

Explain how you know.



Explain how bisecting an angle could lead to the construction of a rhombus. Explain how it could lead to the construction of a kite.



These shapes are drawn very sketchily and are intended to be quadrilaterals. They are marked correctly. What shapes could they be? How did you decide?



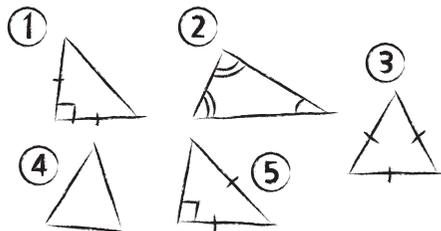
This is an isosceles triangle. It is not drawn to scale.

Pete measures one of the angles in the triangle and says, "This angle is 42 degrees".

Carol measures a different angle in the triangle and says, "This angle is 63 degrees".

Explain how you know that at least one of them is wrong.

The triangles below have been sketched badly, but are labelled correctly.
Which of them are definitely isosceles, and which might be isosceles?
Which one is impossible?
Explain how you know.



GEOMETRY AND MEASURES

Selected National Curriculum Programme of Study statements

Pupils should be taught to:

- Draw and measure line segments and angles in geometric figures
- Apply the properties of angles at a point, angles at a point on a straight line, vertically opposite angles
- Understand and use the relationship between parallel lines and alternate and corresponding angles
- Derive and use the sum of angles in a triangle and use it to deduce the angle sum in any polygon, and to derive properties of regular polygons

The Big Ideas

These statements focus on angles and proof.

Reasoning with angles offers pupils a context through which they can demonstrate their understanding of mathematical proof.

The aim is to enable pupils to construct a logical argument, as well as to use the correct vocabulary (you might, for example, like to challenge a pupil who offers the justification 'because vertically opposite angles are equal' to describe how it is that they know that vertically opposite angles are equal).

The angles context provides some relatively simple facts upon which pupils are able to show that they can construct a more complex argument.

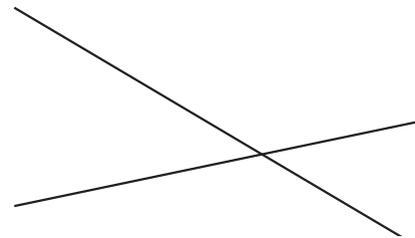
Mastery

1. Draw a line 5cm long
2. At one end of the line mark an angle of 108°
3. Draw a line 5cm long from this end of the line, towards the 108° mark
4. Repeat steps 2 and 3 until your lines join up

Which polygon have you drawn? How do you know?

Mastery with Greater Depth

Four children are practising using a protractor. They each measure a different angle in this diagram.



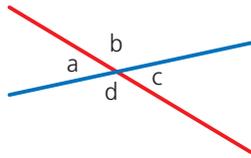
Aled says, "My angle is 42° "
Ben says, "My angle is 40° "
Cassidy says, "My angle is 107° "
Dietmar says, "My angle is 140° "

Without measuring, which one of these children do you think needs help with using a protractor? Explain how you know.

Sam sketches out the diagram below. The angles are labelled A , B , C and D . Without measuring, which of these do you know must be true?

Explain your thinking.

- $a+b = 180$
- $a+c = 180$
- $b+c = 180$
- $b+d = 180$
- $a=b$
- $a=c$
- $b=c$
- $b=d$

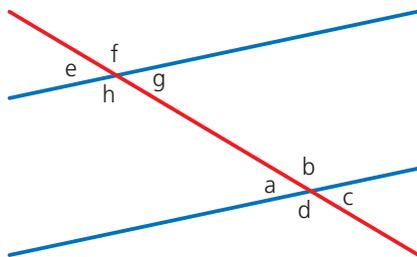


The two blue lines in the diagram below are parallel. The red line crosses both blue lines. The angles are labelled a to h .

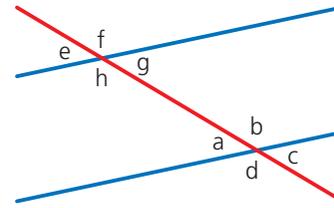
Without measuring, which of these do you know must be true?

Explain your thinking.

- $e=g$
- $e=c$
- $e=h$
- $e+h=180$
- $g+h=180$
- $g+b=180$
- $a+c+f+h=360$
- $a+c+e+g=360$



In this diagram the two blue lines are parallel.



Which angles could be used in the box to give a correct equation?

- $a = \square$
- $180 - a = \square$

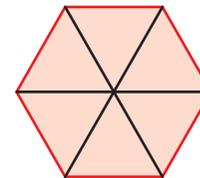
Explain your answers.

If the two blue lines are no longer parallel, which letters can still be used to correctly complete the equations?

Explain your answer.

Six equilateral triangles are arranged to make a regular hexagon.

Sarah says, "The hexagon is made of six triangles. Each triangle's interior angles add to 180° and so the sum of the interior angles of the hexagon is $6 \times 180 = 1080^\circ$ "



Is Sarah right?

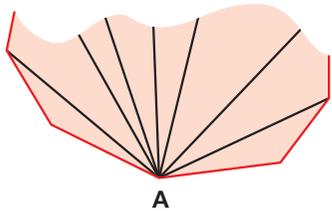
Explain how you know.

The diagram below shows part of a polygon. This polygon had been divided into triangles by joining every vertex to point A.

A part of the polygon has been torn off.

How many edges did the polygon have originally?

Explain how you know.

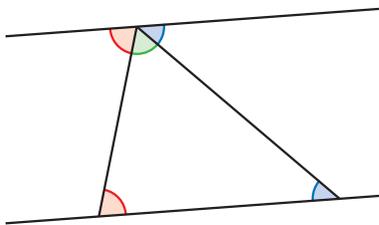


Lucie has drawn a regular polygon.

She measures one of the interior angles as 135° .

Which regular polygon has Lucie drawn?

How do you know?



Use this diagram to prove that the sum of the interior angles in a triangle is 180° .

Lucie has drawn a regular polygon.

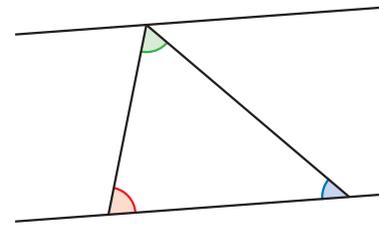
James measures one of the interior angles as 148° .

James' measurement is not accurate.

Which regular polygon do you think Lucie has drawn?

Explain your thinking.

Use this diagram to prove that the sum of the interior angles in a triangle is 180° .



GEOMETRY AND MEASURES

Selected National Curriculum Programme of Study statements

Pupils should be taught to:

- Use the standard conventions for labelling the sides and angles of triangle ABC, and know and use the criteria for congruence of triangles
- Identify properties of, and describe the results of, translations, rotations and reflections applied to given figures
- Identify and construct congruent triangles, and construct similar shapes by enlargement, with and without coordinate grids

The Big Ideas

Variance and invariance are important ideas in mathematics, particularly in geometry, and transformations offer a useful context to explore this. The question 'What's the same?' and 'What's different?' can draw pupils' attention to variance and invariance.

Again, the aim is to enable pupils to construct a logical argument – geometry offers many opportunities for pupils to demonstrate their grasp of deductive reasoning.

Mastery

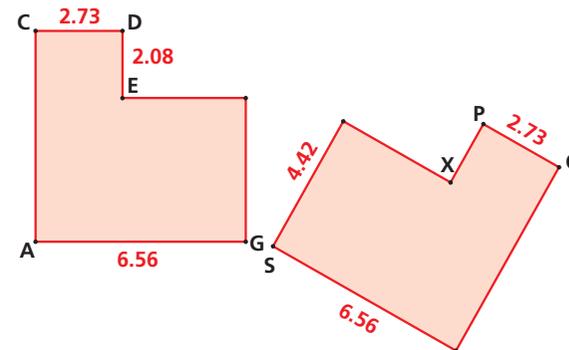
Which of the following must be congruent, which might be congruent, and which are definitely not congruent?

Explain your reasoning in each case.

- Two triangles, both with angles of 30° , 50° and 100° .
- Two triangles, both with sides 3cm, 4cm and 5cm.
- A triangle with sides 6cm, 7cm and 8cm, and a second triangle with sides 7cm, 8cm and 9cm.
- A regular hexagon and a regular pentagon.
- Two equilateral triangles, both with one side that is 10cm long.

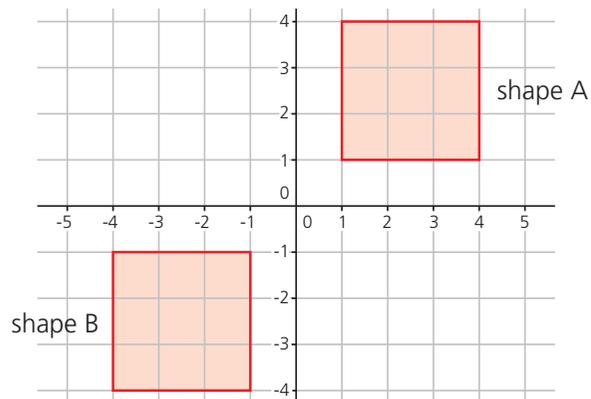
Mastery with Greater Depth

These two shapes are congruent.



All angles are 90° .
How long is AC?
Explain how you know.

Shape **B** is a transformation of shape **A**.



Alex says, "Shape **A** has been reflected to make shape **B**".
 Berenice says, "Shape **A** has been rotated to make shape **B**".
 Claudia says, "Shape **A** has been translated to make shape **B**".
 Deepak says, "Shape **A** has been enlarged to make shape **B**".
 Are any of them correct? If so, who?
 Explain your thinking.

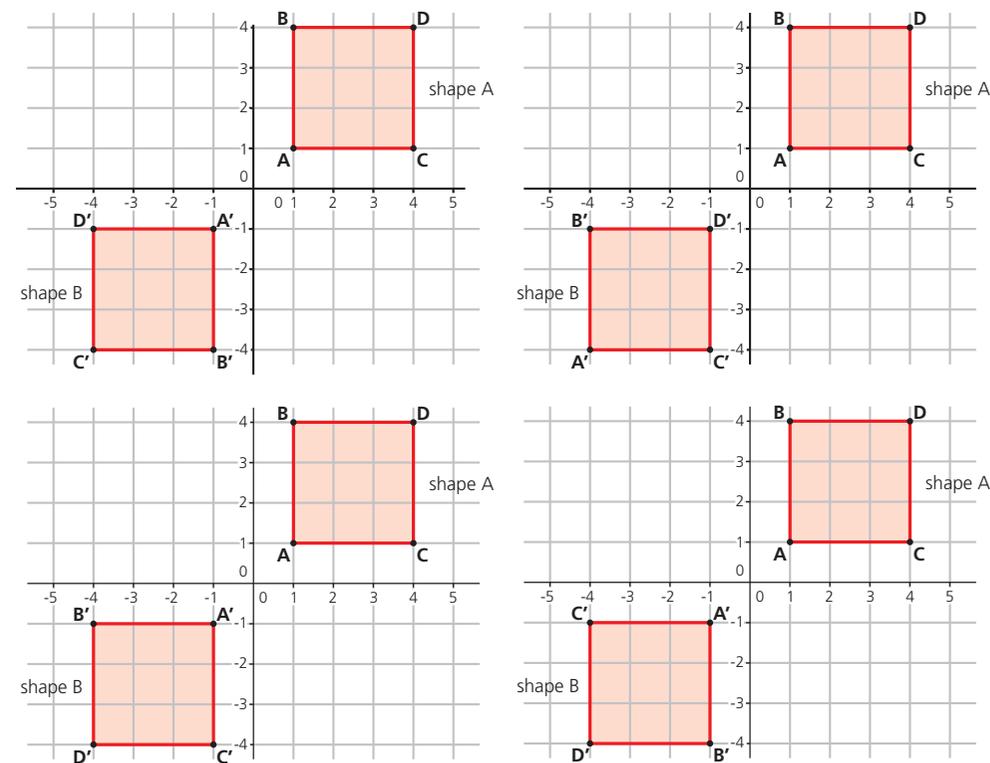
Two shapes are similar, but not congruent.

Which of the following is possible?

- One is a reflection of the other
- One is a rotation of the other
- One is an enlargement of the other
- One is a translation of the other

Explain your reasoning.

For each of the four situations in the diagram below, if possible, describe the single transformation that will move shape **A** onto shape **B**. Look closely at the labelling.



Explain your answers.

Two shapes are congruent.

Which of the following are possible?

- One is a reflection of the other
- One is a rotation of the other
- One is an enlargement of the other
- One is a translation of the other

Explain your reasoning.

Which of the following statements are always true, which are sometimes true and which are never true? Explain your thinking for each statement.

- If a square undergoes an enlargement, the area of the enlargement will be greater than the area of the original square
- Square **A** is similar to Square **B**
- Triangle **A** is similar to Triangle **B**
- Adding 2cm to each side of a square will create an enlargement of the square
- Adding 2cm to each side of a rectangle will create an enlargement of the rectangle
- Adding 2cm to each side of a triangle will create an enlargement of the triangle

PROBABILITY

Selected National Curriculum Programme of Study statements

Pupils should be taught to:

- Record, describe and analyse the frequency of outcomes of simple probability experiments involving randomness, fairness, equally and unequally likely outcomes, using appropriate language and the 0-1 probability scale
- Understand that the probabilities of all possible outcomes sum to 1
- Enumerate sets and unions/intersections of sets systematically, using tables, grids and Venn diagrams
- Generate theoretical sample spaces for single and combined events with equally likely, mutually exclusive outcomes and use these to calculate theoretical probabilities

The Big Ideas

Pupils are first introduced to probability in KS3 (it is no longer included in the primary mathematics programme of study) and so intuitive ideas about risk will already have formed. Jones, Pratt and Watson (2013)¹⁵ state that 'Mathematics teachers at Key Stage 3 need to introduce pupils to the use of probabilistic reasoning by helping them to recognise their naïve or misguided intuitions. Situations that involve uncertainty can be analysed by asking pupils to make predictions and to account for differences between their expectations and what actually happens. Pupils' attention will need to be drawn not only to short-term variation but also to the sense in which the longer-term aggregated view is in fact predictable – at least in a probabilistic sense.'

Mastery

Some counters are placed in a bag.

You win if you pull a red counter out of the bag.

Which of these would you choose?

- Bag A has one red counter and two blue counters
- Bag B has ten red counters and twenty blue counters
- Bag C has 100 red counters and two hundred blue counters

Explain your thinking.

Mastery with Greater Depth

Consider two football teams who have not yet played each other, but who have both won their last ten games.

If they now play each other, which of the following do you agree with?

- Both teams have an equal chance of winning the match.
- The game is likely to be a draw.
- The game is likely to be close but one of the teams will probably win.
- Their previous wins can't tell us anything about what's likely to happen when they play each other.

Explain your thinking.

¹⁵Watson, A., Jones, K. and Pratt, D., 2014. Key Ideas in Teaching Mathematics - Probabilistic Reasoning. NCETM Secondary Magazine 110. <https://www.ncetm.org.uk/resources/44592>

Which of these statements is correct?

Explain your thinking for each statement.

- When flipping a coin there are two possible outcomes (heads or tails) and only one of them can happen. Each of them has a probability of one half.
- In a football match there are three possible outcomes (win, lose or draw) and only one of them can happen. Each of them has a probability of one third.
- When sitting at a table there are four possible seats (one is red, one yellow, one red and one green). You can only sit in one seat. The probability you sit in the green seat is one quarter.
- A 100m race has five runners (Adam, Max, Grace, Emily and Tilly). Each of them has a probability of one fifth of winning.
- When rolling a die there are six numbers. The probability of rolling each of the numbers is one sixth.

Benedict says that he flipped a coin twenty times and got tails every time.

Do you think Benedict is telling the truth?

Explain why.

Sam also flipped a coin 20 times. He got tails 14 times. Sam says, "My coin is biased towards tails." Do you agree with Sam?

Explain why.

A bag contains red, yellow and blue counters.

The probability that a red counter is drawn out of the bag at random is 0.25.

What is the probability that a counter drawn at random is either yellow or blue?

Explain how you know.

Some red, yellow and blue counters are put in a bag and a counter is chosen at random.

- If the number of red counters is doubled, does the probability of selecting a red counter double?
- If the number of red, yellow and blue counters is doubled, does the probability of selecting each colour double?
- If one extra red counter is added, what happens to the probability of selecting a yellow counter?

Explain your reasoning.

Elinor is going on holiday tomorrow.

She checks the weather and sees that it's rained in the resort she's visiting every day for the last week.

Elinor says, "That's great! It means it's less likely to rain while I'm there."

Do you agree with Elinor?

Explain why.

Some counters are placed in a bag.

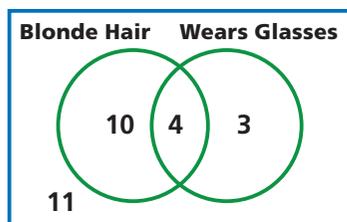
When a counter is drawn at random, the probability that it is not blue is 0.3

There are more than 100 counters in the bag.

How many counters could be in the bag? How many of them would be blue?

Explain your reasoning.

Mrs Holt records on a Venn diagram whether children in her class have blonde hair and whether they wear glasses.



Mr Hall says, "About half of the children with blonde hair also wear glasses."

Mrs Holt says, "About a third of the children with blonde hair also wear glasses."

Who do you agree with? Explain your thinking.

Data about the number of boys and girls in a school are given in a two-way table.

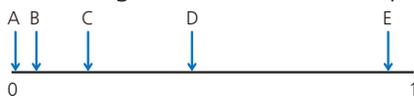
	Boy	Girl
Year 7	89	91
Year 8		

There are fewer students in Year 8 than there are in Year 7, but there are more girls in Year 8 than Year 7.

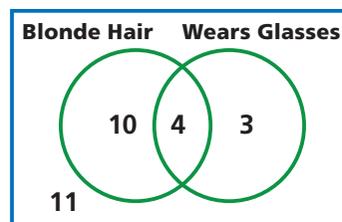
How might you complete the table? Explain your thinking.

Some events are marked on a probability scale.

What might each event be? Explain your thinking.



Mrs Holt records whether children in her class have blonde hair and whether they wear glasses.



Which of the following do you agree with? Explain why.

- It is likely that a child chosen at random from Mrs Holt's class will have blonde hair.
- If a child in Mrs Holt's class has blonde hair, they are likely to wear glasses.
- If a child in Mrs Holt's class wears glasses, they are likely to have blonde hair.

What other true statements can you make based on the Venn diagram?

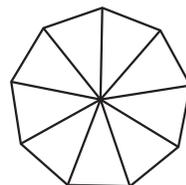
How many events can you name that have a probability of exactly one half?

How many have a probability of exactly one quarter?

How many have a probability of exactly three quarters?

Explain your thinking.

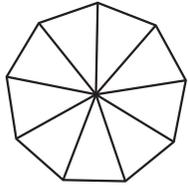
This is a template for a spinner for a board game:



Use four colours to shade it so that:

- The probability of spinning red is as close to $\frac{2}{5}$ as possible.
- The probability of spinning green is less than $\frac{1}{3}$.
- The probability of spinning yellow is less than the probability of spinning blue.

This is a template for a spinner for a board game:



Use four colours to shade this spinner so that:

- Red is more likely than blue.
- The probability of spinning green is at least $\frac{1}{3}$.
- The probability of spinning yellow is exactly $\frac{2}{9}$.

Two dice are rolled and the totals are multiplied.
Player A wins if the resulting product is more than 12.
Player B wins if the resulting product is less than 12.
If the product is exactly 12, the game is a draw.
Would you choose to be player A or player B?
Explain why.

STATISTICS

Selected National Curriculum Programme of Study statements

Pupils should be taught to:

- Describe, interpret and compare observed distributions of a single variable through: appropriate graphical representation involving discrete, continuous and grouped data; and appropriate measures of central tendency (mean, mode, median) and spread (range, consideration of outliers)
- Construct and interpret appropriate tables, charts, and diagrams, including frequency tables, bar charts, pie charts, and pictograms for categorical data, and vertical line (or bar) charts for ungrouped and grouped numerical data

The Big Ideas

The KS3 Programme of Study states that pupils should be taught to 'explore what can and cannot be inferred in statistical and probabilistic settings, and begin to express their arguments formally.' One aim is to enable pupils to calculate measures of spread, and draw accurate charts and graphs. Another is to enable pupils to interpret the graphs and the statistics that they have calculated (or have been given) and use these summary figures to draw conclusions or explore further lines of enquiry.

Mastery

Write down five integers with a mean of 6
...and a median of 4
...and a mode of 2
...and a range of 13
Justify your answers.

From 7th March 2016 to 7th March 2017 a swimming club had the same members.
Complete the table to show the information about the members of the club.
Explain your reasoning.

Age of member	
Mean (March 2016)	14 years 7 months
Range (March 2016)	4 years 2 months
Mean (March 2017)	
Range (March 2017)	

(QCA, 2002)

Mastery with Greater Depth

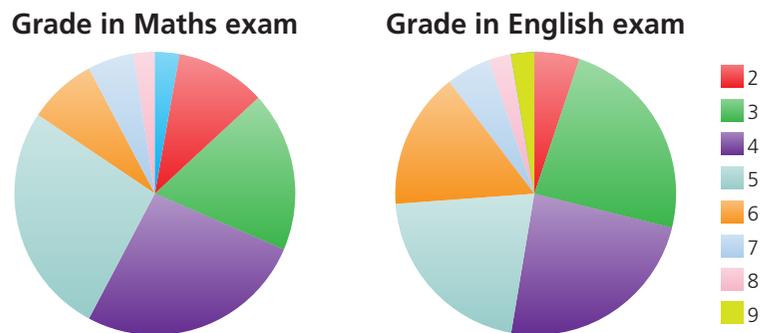
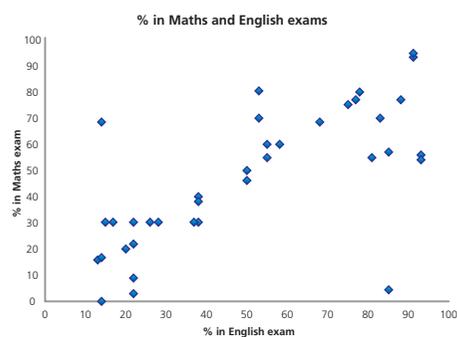
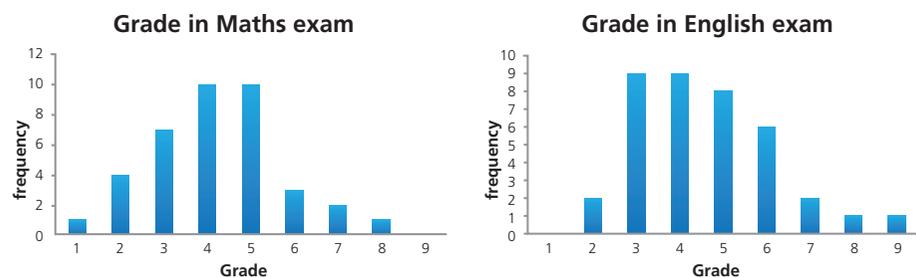
Ten friends work in jobs where they each earn around the national average salary.
One of them then gets a new job that pays a salary of £10 million per year.
Describe how this changes the 'average' salary for the group of friends.

Five numbers have a mean of 10.
Three of the numbers are 10, 10 and 10.
What can you say about the other two numbers in the data set?

Justine is exploring the hypothesis 'students who are good at English are also good at maths.'

She has data for some of the students in her school showing how well they scored in their maths exam and their English exam.

Justine draws these graphs. Comment on what each graph shows and whether it helps Justine in exploring her hypothesis.



A business owner wants to compare the footfall at three shops.

For a month she gathers data about the number of customers, their gender and how much they spend at each of the shops.

- To compare the number of customers as each shop, should she use a bar chart, pie chart or line graph? Explain your reasoning.
- To compare the amount of money taken at each shop through the month, should she use a bar chart, pie chart or line graph? Explain your reasoning.
- To compare the gender split for the customers at each shop, should she use a bar chart, pie chart or a line graph? Explain your reasoning.

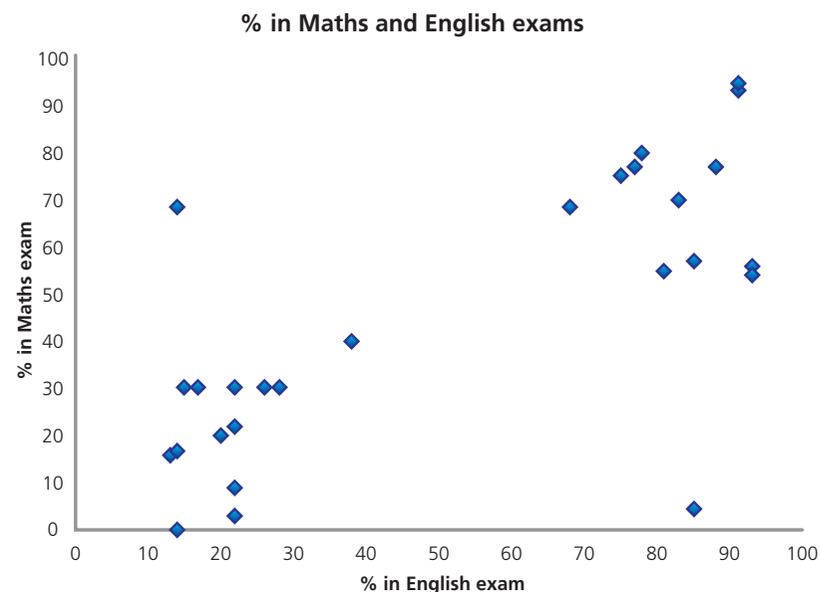
Justine is exploring the hypothesis 'students who are good at maths are also good at English.'

She gathers data and draws a scatter graph.

What does this graph show?

Does it help Justine to see if her hypothesis is correct?

Are there any results that you'd like to explore further or any other questions that the scatter graph raises for you?



Two classes are each given a test. A brother and sister, Luke and Jade, are each in one of the classes.

They draw pie charts for the results of each class. Luke says, "There were more grade sevens in my class than in yours."

Is Luke correct? Explain your thinking.

Luke's Class



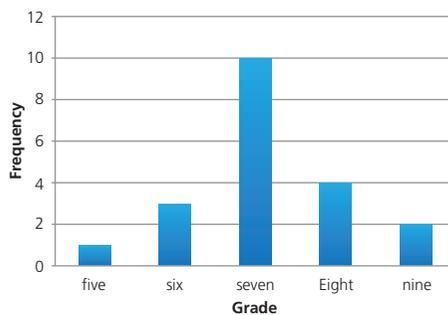
Jade's Class



Jade draws bar charts for the results of each class. She says, "There were more grade sevens in my class than in yours."

Is Jade correct? Explain your thinking.

Luke's Class



Jade's Class

