MathsHUBS Resources

National Centre for Excellence in the Teaching of Mathematics

Mastery Professional Development

3 Multiplicative reasoning

3.1 Understanding multiplicative relationships

Guidance document | Key Stage 3

Making connections

The NCETM has identified a set of six 'mathematical themes' within Key Stage 3 mathematics that bring together a group of 'core concepts'.

The third of these themes is *Multiplicative reasoning*, which covers the following interconnected core concepts:

3.1 **Understanding multiplicative relationships**

3.2 Trigonometry

This guidance document breaks down core concept *3.1 Understanding multiplicative relationships* into six statements of knowledge, skills and understanding:

3.1.1 Understand the concept of multiplicative relationships

3.1.2 Understand that multiplicative relationships can be represented in a number of ways and connect and move between those different representations

3.1.3 Understand that fractions are an example of a multiplicative relationship and apply this understanding to a range of contexts

3.1.4 Understand that ratios are an example of a multiplicative relationship and apply this understanding to a range of contexts

3.1.5 Understand that percentages are an example of a multiplicative relationship and apply this understanding to a range of contexts

3.1.6 Understand proportionality

Then, for each of these statements of knowledge, skills and understanding we offer a set of key ideas to help guide teacher planning.

Please note that these materials are principally for professional development purposes. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

Overview

Multiplicative relationships underpin many aspects of mathematics at Key Stage 3, but students often experience them as distinct topics with no obvious connections. Percentages, fractions, proportionality and ratio, for example, can all be considered as contexts in which multiplicative relationships are used and explored. It is, therefore, important that the vocabulary and imagery used in all contexts is consistent in order to support students in their understanding that the same mathematical principles are involved. In many cases there will be several different possible representations that could be used to help understand the mathematical structure of a situation. An important aspect of work with students will be to consider the relative usefulness and efficiency of different representations and approaches.

Students will have met fractions and percentages at Key Stage 2, while proportionality and ratio may be new to them. A key idea in Key Stage 3 will be to connect fractions, percentages, proportionality and ratio together through the overarching idea of multiplicative relationships.

Students should have interpreted multiplication as scaling at Key Stage 2, but here it is developed in more depth. Students should recognise that it is possible to go from any number* to any other number by multiplying, and not simply view multiplication as repeated addition, because this could lead to incorrect additive strategies. For example, in the question *'If five miles is equivalent to eight kilometres, how many kilometres is seven miles?'* students may respond that two more miles means two more kilometres. Students should experience multiplicative relationships in many different contexts in order to overcome such errors.

Exploring a range of real-life contexts (including use of compound measures) will further support students' understanding of proportionality. Stressing the notion that when one measure doubles (or trebles or is multiplied by any scale factor) so too does the other can usefully highlight the terminology of 'direct' proportion and this can be contrasted with inverse proportion, which is a key idea to introduce at Key Stage 3.

*except the specific case involving zero as one of the factors but not the product.

Prior learning

Before beginning to teach *Understanding multiplicative relationships* at Key Stage 3, students should already have a secure understanding of the following from previous study:

Key stage	Learning outcome
Upper Key Stage 2	• Multiply proper fractions and mixed numbers by whole numbers, supported by materials and diagrams • Recognise the per cent symbol (%) and understand that per cent relates to 'number of parts per hundred', and write percentages as a fraction with denominator 100, and as a decimal • Use all four operations to solve problems involving measure (for example, length, mass, volume, money) using decimal notation, including scaling • Solve problems involving the relative sizes of two quantities where missing values can be found by using integer multiplication and division facts

	• Solve problems involving the calculation of percentages (for example, of measures, and such as 15% of 360) and the use of percentages for comparison • Solve problems involving similar shapes where the scale factor is known or can be found • Solve problems involving unequal sharing and grouping, using knowledge of fractions and multiples

You may find it useful to speak to your partner schools to see how the above has been covered and the language used.

You can find further details regarding prior learning in the following segments of the NCETM primary mastery professional development materials[1]:

- Year 4: 2.17 Structures: using measures and comparison to understand scaling
- Year 6: 2.27 Scale factors, ratio and proportional reasoning
- Year 6: 3.10 Linking fractions, decimals and percentages

Checking prior learning

The following activities from the NCETM primary assessment materials[2] offer useful ideas for assessment, which you can use in your classes to check whether prior learning is secure:

Reference	Activity
Year 6 page 21	*Last month Kira saved* $\frac{3}{5}$ *of her £10 pocket money. She also saved 15% of her £20 birthday money.* *How much did she save altogether?*
Year 6 page 24	*If I share equally a 3 m ribbon between five people, how long will each person's ribbon be?*
Year 6 page 25	*Sam and Tom share 45 marbles in the ratio 2 : 3.* *How many more marbles does Tom have than Sam?*

Key vocabulary

Term	Definition
proportion	1. A part-to-whole comparison. Example: Where £20 is shared between two people in the ratio 3 : 5, the first receives £7.50, which is $\frac{3}{8}$ of the whole £20. This is the first person's proportion of the whole. 2. If two variables x and y are related by an equation of the form $y = kx$, then y is directly proportional to x; it may also be said that y varies directly as x. When y is plotted against x, this produces a straight line graph through the origin.

	3. If two variables x and y are related by an equation of the form $xy = k$, or equivalently $y = \frac{k}{x}$, where k is a constant and $x \neq 0$, $y \neq 0$, they vary in inverse proportion to each other.
ratio	A part-to-part comparison. The ratio of a to b is usually written $a:b$. Example: In a recipe for pastry, fat and flour are mixed in the ratio $1:2$, which means that the fat used has half the mass of the flour. That is, $\frac{\text{amount of fat}}{\text{amount of flour}} = \frac{1}{2}$. Thus, ratios are equivalent to particular fractional parts.

Collaborative planning

Below we break down each of the six statements within *Understanding multiplicative relationships* into a set of key ideas to support more detailed discussion and planning within your department. You may choose to break them down differently depending on the needs of your students and timetabling; however, we hope that our suggestions help you and your colleagues to focus your teaching on the key points and avoid conflating too many ideas.

Please note: We make no suggestion that each key idea represents a lesson. Rather, the 'fine-grained' distinctions we offer are intended to help you think about the learning journey irrespective of the number of lessons taught. Not all key ideas are equal in length and the amount of classroom time required for them to be mastered will vary, but each is a noteworthy contribution to the statement of knowledge, skills and understanding with which it is associated.

The following letters draw attention to particular features:

D Suggested opportunities for **deepening** students' understanding through encouraging mathematical thinking.

L Examples of shared use of **language** that can help students to understand the structure of the mathematics. For example, sentences that all students might say together and be encouraged to use individually in their talk and their thinking to support their understanding (for example, *'The smaller the denominator, the bigger the fraction.'*).

R Suggestions for use of **representations** that support students in developing conceptual understanding as well as procedural fluency.

V Examples of the use of **variation** to draw students' attention to the important points and help them to see the mathematical structures and relationships.

PD Suggestions of questions and prompts that you can use to support a **professional development** session.

For selected key ideas, marked with an asterisk (*), we exemplify the common difficulties and misconceptions that students might have and include elements of what teaching for mastery may look like. We provide examples of possible student tasks and teaching approaches, together with suggestions and prompts to support professional development and collaborative planning. You can find these at the end of the set of key ideas.

Key ideas

3.1.1 Understand the concept of multiplicative relationships

That any two quantities can be linked multiplicatively is a key awareness in this core concept. It will be important for students to see that multiplication is more than simply repeated addition. While students may recognise that, for example, three and six can be connected via a multiplication, it is much less likely that they will see three and five as being linked multiplicatively.

Offering students experiences that encourage them to look for multiplication rather than addition will support students in developing this more sophisticated view of multiplication (i.e. scaling, which they will have been introduced to in Key Stage 2). It will also lead to the important realisation that any two numbers can be linked in this way.

Later, students will appreciate that the multiplier can be expressed as the fraction comprising the two numbers (for example, $3 \times \frac{5}{3} = 5$ and $5 \times \frac{3}{5} = 3$).

3.1.1.1* Appreciate that any two numbers can be connected via a multiplicative relationship

3.1.1.2 Understand that a multiplicative relationship can be expressed as a ratio and as a fraction

3.1.1.3 Be able to calculate the multiplier for any given two numbers

3.1.1.4 Appreciate that there are an infinite number of pairs of numbers for any given multiplicative relationship (equivalence)

3.1.2 Understand that multiplicative relationships can be represented in a number of ways and connect and move between those different representations

This collection of key ideas explores some of the images and representations (for example, double number track, ratio table, double number line (also known as a *stacked number line*), scaling diagram, graphs, algebraic symbolism and other notation) that can be used to build an understanding of the different interpretations of multiplicative structures and so make the connections between seemingly distinct topics explicit. It is important to keep in mind that the purpose of these different representations is to reveal the underpinning mathematical structure, rather than to provide a method to achieve an answer.

3.1.2.1* Use a double number line to represent a multiplicative relationship and connect to other known representations

3.1.2.2* Understand the language and notation of ratio and use a ratio table to represent a multiplicative relationship and connect to other known representations

3.1.2.3 Use a graph to represent a multiplicative relationship and connect to other known representations

3.1.2.4 Use a scaling diagram to represent a multiplicative relationship and connect to other known representations

3.1.3 Understand that fractions are an example of a multiplicative relationship and apply this understanding to a range of contexts

This collection of key ideas explores contexts in which fractions are used to describe and explore a given situation.

Fraction notation holds within it a multiplicative relationship. In a fraction, such as $\frac{2}{3}$, the numerator will be two-thirds of the denominator, and the denominator will be three halves of the numerator. However, a particular focus in these key ideas is the use of a fraction as a multiplier. Students should view a relationship of the form $ab = c$ (where a and/or b is a fraction) from different perspectives and in different contexts.

3.1.3.1 Find a fraction of a given amount

3.1.3.2 Given a fraction and the result, find the original amount

3.1.3.3 Express one number as a fraction of another

3.1.4 Understand that ratios are an example of a multiplicative relationship and apply this understanding to a range of contexts

Here, ratios are used to describe and explore multiplicative relationships. Students may still view some of these contexts additively, but it is vital that the multiplicative aspect is explored and emphasised. For example, in a question such as *'Some money is shared between Alan and Layla in the ratio 2 : 3. If Alan receives £10, how much does Layla receive?'* students may perceive an entirely additive structure: divide the whole into five parts, take two parts for Alan and three parts for Layla. However, students should also have the awareness that Alan and Layla's money is linked by multiplicative relationships: Layla has $\frac{3}{2}$ of Alan's share, and Alan has $\frac{2}{3}$ of Layla's share. Also, Alan has $\frac{2}{5}$ of the total and Layla has $\frac{3}{5}$ of the total.

The idea of a rate then becomes an integral part of this multiplicative relationship. The ratio 2 : 3 can be thought of as 'for every £2 Alan has, Layla has £3' or, by considering the multiplier, 'for every £1 Alan has, Layla has £1.50'.

The double number line and ratio table representations can be key in supporting students to see a rate as representing different multiplicative relationships. It is important to emphasise that there are two multiplicative relationships evident in each situation: one which scales one value or quantity to the next (for example, if £3 = \$4, then £6 = \$8, doubling each value), and one which converts one value or quantity to another (if £3 = \$4, then £7.50 = \$10 since 7.5 is $\frac{3}{4}$ of 10).

3.1.4.1 Be able to divide a quantity into a given ratio

3.1.4.2 Be able to determine the whole, given one part and the ratio

3.1.4.3* Be able to determine one part, given the other part and the ratio

3.1.4.4 Use ratio to describe rates (e.g. exchange rates, conversions, cogs, etc.)

3.1.5 Understand that percentages are an example of a multiplicative relationship and apply this understanding to a range of contexts

In this set of key ideas, the use of percentages to represent multiplicative relationships is explored.

Students may use informal additive methods to calculate percentages. For example, to find 16% of a total they will find 10%, find 5%, find 1% and add these together. While it is important for students to know this, and to be able to work flexibly with percentages, it is also important for efficiency and depth of understanding that they recognise them as multiplicative relationships and understand that there exists a single multiplier linked to any percentage.

As with ratio, the double number line and ratio table representations are useful in supporting students in identifying and working with the multiplier, and consistent use of these representations through ratio, percentages and proportion may make the connections between these apparently different topics more obvious for students.

Throughout 3.1.5 students are again working on the relationship $ab = c$, where a or b is written as, or interpreted as, a percentage, and are exploring this in different contexts and with different representations.

3.1.5.1 Describe one number as a percentage of another

3.1.5.2 Find a percentage of a quantity using a multiplier

3.1.5.3* Calculate percentage changes (increases and decreases)

3.1.5.4 Calculate the original value, given the final value after a stated percentage increase or decrease

3.1.5.5 Find the percentage increase or decrease, given start and finish quantities

3.1.6 Understand proportionality

As students' understanding of multiplicative relationships matures, they can work with more complex contexts and use algebraic notation to generalise. An important awareness here is that there is one unifying structure which connects fractions, percentages and ratio, and that this one structure can be described by the algebraic formulae $x \times k = y$ or alternatively $k = \frac{y}{x}$, where x and y are the quantities in proportion and k is the constant of proportionality. While exploring a wide range of examples of proportionality (including examples of *'what it's not'*) it will be important to make the distinction between linear relationships which are not proportional (i.e. of the form $y = mx + c$ rather than $y = kx$) and also to become aware of situations where the variables are inversely proportional (i.e. $y = k \times \frac{1}{x}$ or $y = \frac{k}{x}$). In formalising this generalisation, students are able to use the underlying structure to develop an awareness that there are different types of proportionality, particularly inverse proportionality.

3.1.6.1 Understand the connection between multiplicative relationships and direct proportion

3.1.6.2 Recognise direct proportion and use in a range of contexts, including compound measures

3.1.6.3 Recognise and use inverse proportionality in a range of contexts

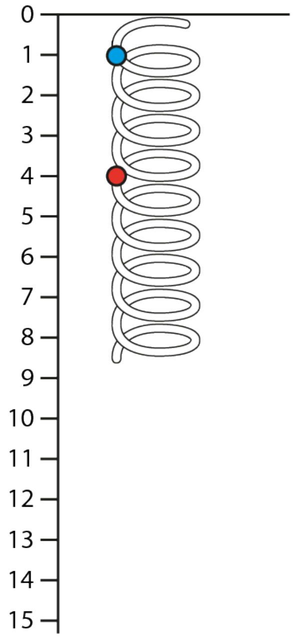
Exemplified key ideas

3.1.1.1 Appreciate that any two numbers can be connected via a multiplicative relationship

Common difficulties and misconceptions
Students often assume that relationships are additive, including in situations where the actual relationship is multiplicative (for example, in scaling and other proportional situations). Because students may see multiplication as only repeated addition (as opposed to scaling) their methods often entail building up to an answer by dealing with small segments of the problem and then adding the answers together. When this is not possible, it is not uncommon for students to consider the difference $a - b$ rather than the ratio a : b. It is important that students experience a range of contexts and are encouraged to discern between additive and multiplicative situations.

What students need to understand	Guidance, discussion points and prompts
Interpret multiplication as scaling. *Example 1:* *Put the following products in order of size from smallest to largest.* a) $1\,142 \times 2$ b) $1\,142 \times \frac{19}{9}$ c) $\frac{7}{9} \times 1\,142$ d) $1\,142 \times \frac{9}{7}$ e) $1\,142 \times \frac{17}{9}$ f) $2\,284 \times \frac{4}{9}$	**V** In *Example 1*, the number 1 142 is chosen so that students are not tempted to calculate. Rather, they should be encouraged to reason about the magnitude of the product. **L** The numbers are chosen to encourage the use of language such as 'two times as big' and to help students make comparisons relative to that value. **PD** Discuss the way that language may influence the image of multiplication for students. Do they have a different picture in their heads for '*three groups of four*' and '*three times as large as four*'? How is the picture different?

Example 2:



a) *The spring is stretched so that the blue dot aligns with the 2 cm mark. Where is the red dot?*

b) *The spring is stretched so that the red dot is on 6 cm. Where is the blue dot?*

c) *The spring is compressed so that the red dot is on 3 cm. Where is the blue dot?*

d) *At the moment the two dots are 3 cm apart. Where is each dot if they are 12 cm apart?*

R The use of a dynamic context, such as the one in *Example 2*, can offer an image that shows the continuous nature of multiplication. In *Example 2*, a spring is hung next to a ruler. A blue dot is marked on the spring at 1 cm, and a red dot at 4 cm. Students will need time to understand the representation. You might start by asking them to decide whether the two dots will always be the same distance apart as the spring is stretched, before working on the questions posed.

PD What type of questions could you ask to ensure students' understanding? You could consider asking questions such as:

- *'How do you picture the two dots on the spring as the spring is stretched?'*
- *'How did you tackle the questions?'*
- *'Which of the four questions did you find the most challenging to picture?'*
- *'Did this image encourage you to visualise the operation of multiplication in a different way? If so, how?'*

Example 3:

×	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	18	27	36	45	54	63	72	81	90	99	108
10	10	20	30	40	50	60	70	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

a) Look at the 4s row. What does 4 need to be multiplied by:

(i) to move to 8?

(ii) to move to 28?

(iii) to move to 16?

(iv) to move to 20?

b) Still within the 4s row, what does the 8 need to be multiplied by:

(i) to move to 16?

(ii) to move to 24?

(iii) to move to 20?

(iv) to move to 4?

c) Consider corresponding entries in the 4s row and the 6s row.

(i) What is the relationship between 8 and 12; 20 and 30; 40 and 60, etc.?

(ii) Picture where 10 might be in the 4s row. What number would be in the corresponding position in the 6s row?

R You could use a multiplication square to explore multiplicative relationships, as in *Example 3*. Here, the number 4 has been chosen as a starting number. Draw students' attention to the fact that multiplication by a non-integer can 'fill the gaps' when moving from one number to another multiplicatively. This will support students with the realisation that *any* two numbers can be linked multiplicatively.

If students are familiar with the spring representation for multiplication, as shown in *Example 2*, then this can also be used to support understanding that multiplication transforms any number onto any other, since a point on the spring can be imagined passing through all points on the number line as the spring is stretched or compressed.

Example 4:

Which sets of numbers show a proportional relationship? Explain your choices.

a)

$\frac{1}{2}$	1
3	6

b)

1	1.1
2	2.2

c)

3	4
7	8

d)

2	4
6	8

e)

8	4
16	8

f)

4	3
1	0.75

V *Example 4* uses ratio tables to provide some examples and non-examples of proportional relationships. The numbers have been selected to offer fractions and decimals, and to show proportional relationships where the value is increasing (as read from left to right) and decreasing.

D An addition to this task might be to select one of the tables that students correctly identify as being proportional, and to then rewrite it, swapping the numbers, asking if the relationship is still proportional.

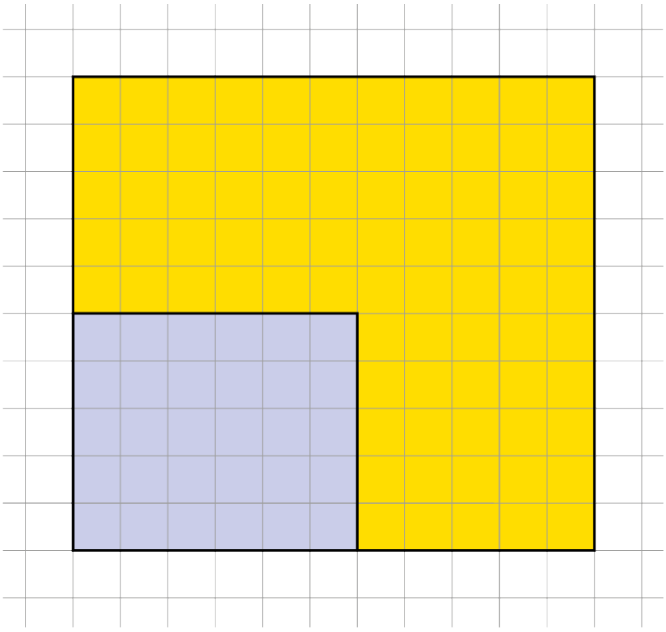
For example, part a) might be rewritten as:

6	3
1	$\frac{1}{2}$

Example 5:

Are these rectangles similar shapes?

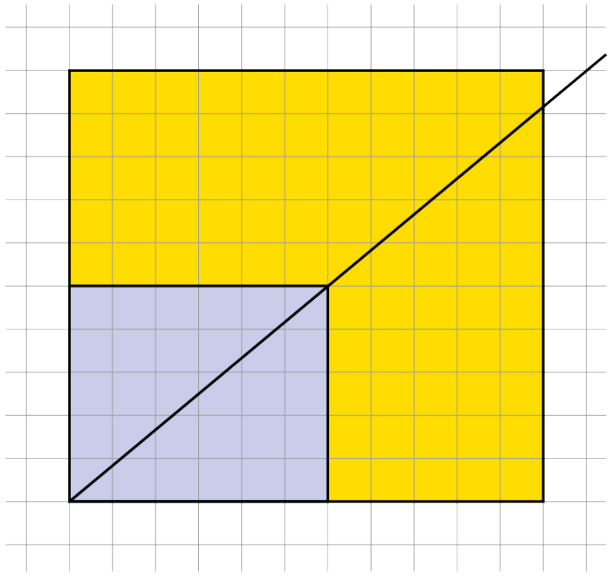
How do you know?



D Given how common the misconception of adding as opposed to multiplying when working with a proportional relationship is, it can be useful to offer a task in which this misconception can be exposed. *Example 5* gives students a (potentially) dynamic geometric context in which to discuss relationships. In this example, the rectangles have been chosen so that the larger one looks like it might be an enlargement of the smaller one; only on closer inspection can it be seen that the rule 'add five to each side' has been used.

The use of a dynamic geometry program (or even just a drawing tool in a word processor) can add to the scaling narrative when exploring this problem with students.

R Drawing a diagonal through the smaller rectangle (or inviting students to do so) might help them to see that the rectangles are not similar.

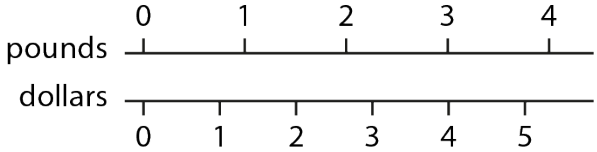


This will support students' understanding later on that, when numbers that are in the same ratio are plotted as x and y coordinates, they will lie on the line $y = mx$, where m has the value of the multiplier.

Interpret multiplication as a rate.

Example 6:

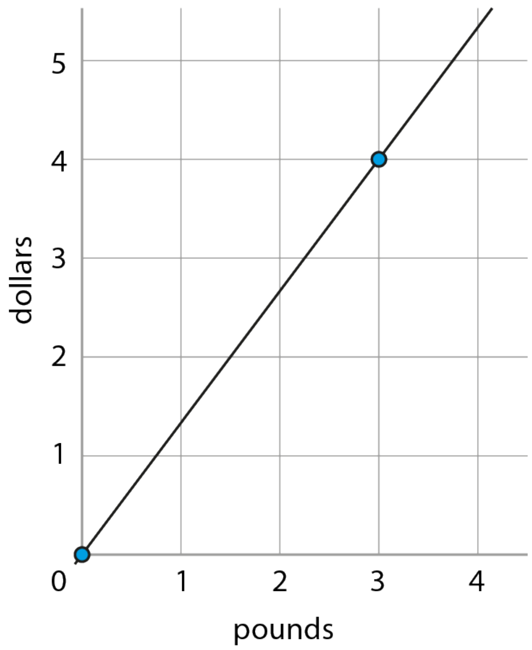
a) *This double number line can be used to convert between pounds and dollars. £3 is equivalent to $4.*



(i) *Describe how you would use the double number line to roughly convert $4.50 to pounds.*

(ii) *Describe how you would use the double number line to accurately convert $4.50 to pounds.*

b) *This graph can be used to convert between pounds and dollars.*



(i) *What features of the graph show that the rate of exchange is £3 for every $4?*

(ii) *What features of the double number line in part a) show that the rate of exchange is £3 for every $4?*

(iii) *Describe how the graph would change if the rate of exchange changed to £3 for every $5.*

(iv) *Describe how the double number line would change if the rate of exchange changed to £3 for every $5.*

R The use of a rate for multiplication provides a context in which some different representations can be introduced and explored. The double number line is introduced in *Example 6* to convert between pounds and dollars. Ensure that students are given enough time to familiarise themselves with this representation.

PD If the double number line is unfamiliar to students, you might like to take time to consider contexts and relationships in which it is a useful representation to work with. How, for example, could you use a double number line to show a 25% reduction in an item's price if it was originally £30? What about if the sale price is £30 and you are asked to find the original price?

Consider the numbers used in *Example 6*. What are the benefits of using £3 ≈ $4 in giving students access to the structure of the double number line? What impact, if any, do you think the use of, for example, £6 ≈ €7 would make on the task?

R The same rate from part a) can be represented using a graph, as shown in part b). The same questions can be asked to encourage students to explore the use of the graph in the context of a rate.

L The idea of rate can be reinforced in part b) by drawing students' attention to the fact that they would receive $4 for every £3. Stressing the 'for every' part of this statement is useful in providing an image for students to work with. This language could be explored further by asking students to state other equivalent quantities of dollars and pounds that they know would align if the two number lines or the graph axes were extended.

It is important to allow students to see the connections between the language and the two graphical representations. Considering what features the different representations have in common, and how these might

	change in a different situation, may help to make these connections more explicit to students. **R** Applying the same numbers and relationship to a different context may help students to further understand rate. For example, you might discuss a journey that was made at the rate of three miles every four minutes and consider how this would look on a double number line and a Cartesian graph. Are students able to find another context where the same relationship applies?
Understand that any number can be transformed into any other number by multiplying. *Example 7:* *Fill in the missing numbers.* a) $5 \times \frac{4}{5} = \square$ b) $5 \times \frac{6}{5} = \square$ c) $5 \times \frac{5}{5} = \square$ d) $5 \times \frac{23}{5} = \square$ e) $5 \times \frac{\square}{5} = 12$ f) $5 \times \frac{17}{\square} = \square$ g) $5 \times \frac{\square}{\square} = 14$ h) $9 \times \frac{\square}{\square} = 14$ i) $17 \times \frac{\square}{\square} = 12$ j) $10 \times \frac{\square}{\square} = 0$ k) $0 \times \frac{\square}{\square} = 10$	**V** *Example 7* has been sequenced to prompt discussion about the size of the product. In parts a) to d), initial discussion could focus on whether the product is less than, greater than or equal to five, so it may be useful to initially display only these parts. Further discussion may usefully focus on the actual product, with students' attention drawn to the connection between the numerator and the product. It will be important to prompt students to reason why this is true and not to be content with just spotting a pattern, but to appreciate the mathematical structure which is producing the pattern. Part c) is of significance since it is the identity function and, while students do not necessarily need to know this terminology, they should understand the significance of a multiplication in which the multiplier and product are equal. Part e) onwards offers an opportunity for students to use what they know to make informed choices without calculating, and to justify these choices. It is worth noting that part f) has multiple possible solutions. Should students question this, you might like to use the opportunity to discuss what numbers they can easily use to fill the gaps and how they know their answers to be correct. While parts g) to i) do not have multiple solutions, the solution can be written in

	different ways and attention should be drawn to the use of an appropriate denominator to allow an answer to be found without calculating. Parts j) and k) draw attention to the cases when zero is either a factor or a product. Part k) will warrant discussion as the only special case where there is no solution. After working on an appropriate number of calculations, and maybe with students writing some of their own, the question *'Is it always possible to go from one number to another by multiplying?'* might be asked and treated as a discussion point.

3.1.2.1 Use a double number line to represent a multiplicative relationship and connect to other known representations

<table>
<tr><th>Common difficulties and misconceptions</th></tr>
<tr><td>
Students may not understand that both lines in a double number line must start at zero, and that these zero marks must be aligned.
The double number line can be a useful image to support students in their understanding of the underlying mathematical structure of a multiplicative relationship. It may not always be an efficient representation with which to calculate an answer, but it is an important representation to think with.
A ratio table, which can be considered to be a compressed version of the double number line, is likely to be more efficient, but in the compression, some of the structure is lost and this may need to be made clear to students who are keen to rush to an answer without thinking.
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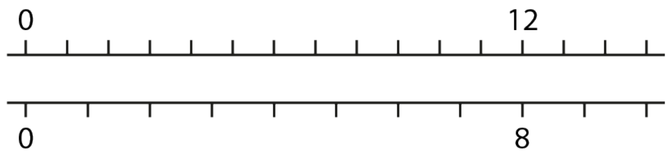
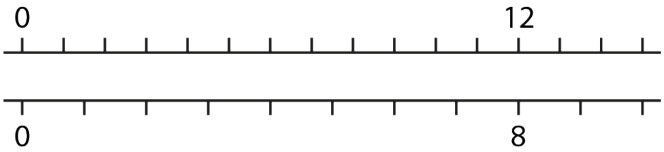
<table>
<tr><th>What students need to understand</th><th>Guidance, discussion points and prompts</th></tr>
<tr><td>
Use the multiplicative nature of the double number line to find pairs of values using scalar (along the line) multipliers.
Example 1:
Ellie and her dad walk side by side along a straight path. The number of steps they take is represented here:



c) Which line represents Ellie's steps and which represents her dad's? Explain how you know.
d) When Ellie had taken 12 steps, her dad had taken 8 steps. When else were Ellie and her dad stepping at precisely the same time?
e) If they continued walking like this, find more points when they would be stepping at the same time. What is the connection between the number of Ellie's steps and the number of her dad's steps?
</td><td>
Students should begin to understand how the double number line represents multiplicative relationships. The numbers in Examples 1– 3 have been chosen so that the functional relationship is relatively straightforward.
PD Consider Example 1 and discuss what students might do in order to find some pairs. What prompts might you use to facilitate further pairs to be found?
R The advantage of the double number line over other representations is that it offers a sense of scale. You could use this by encouraging students to estimate values before going on to calculate accurately.
</td></tr>
</table>

Example 2:

Ali buys eight identical packets of cakes for a birthday party. The total cost is £12.



a) Which line in this double number line represents the number of packets of cakes Ali buys and which represents the total cost? Explain how you know.

b) Use this diagram to write down the cost of some other quantities of packets of cakes.

c) Can you write down roughly how much nine packets of cakes would cost? Can you be precise? Explain how.

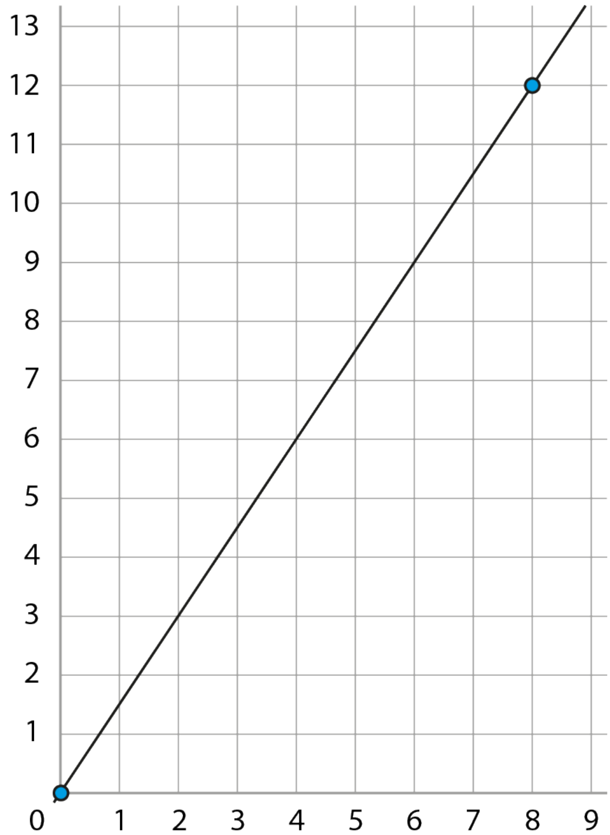
D
R

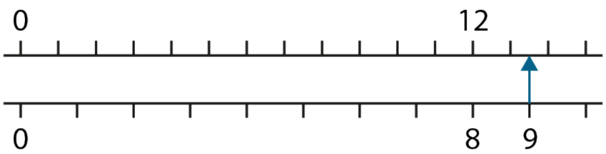
It is important that students make connections between the double number line and other familiar representations in order to see the relationships between them. You could ask students to construct a ratio table or graph, using the information presented in a double number line, to solve a problem.

For example, *Example 2* could be constructed as a ratio table:

12	
8	9

or as a line graph:



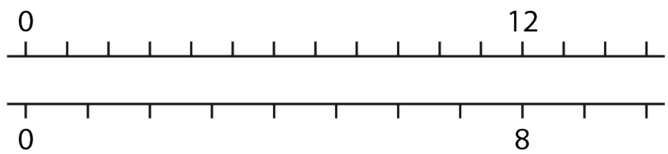


By comparing the double number line and the line graph, students will see that every pair of numbers that align on the double number line are represented by coordinates on the graph. It may be helpful to draw students' attention to the fact that, if the top number line is rotated through 90° anticlockwise the graph is obtained.

PD You could assess students' comprehension by asking them to reflect on the ways they worked on *Example 2*. You might consider asking a question such as *'Did you work by scaling along the lines, or by identifying the relationship between them?'*

Example 3:

Jodie is travelling from America to New Zealand. At the time of travelling US$8 is worth about NZ$12.



d) Which line in this double number line represents US$ and which represents NZ$?

e) Use this diagram to write down some other conversions for Jodie.

(i) How many NZ$ is US$9 worth?

(ii) How many US$ is NZ$9 worth?

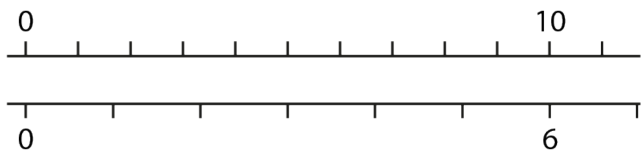
f) Can you continue the lines? What rule are you using?

V In *Examples 1–3*, the same representation is used to focus attention on different contexts in which the same multiplicative relationship is used, and to give students opportunities to work on the double number line in different ways. In *Example 1*, the marks represent steps (a discrete measure) and so the question *'How many steps has Ellie taken when her dad has taken three?'* does not make intuitive sense. However, *Example 2* allows the question of *'How much would three packets of cakes cost?'*, permitting students to use the representation to make an estimate that they can then go on to justify. *Example 3* allows students to work sensibly by moving up and down both lines, and working between them, to convert amounts given in US$ to NZ$ and amounts given in NZ$ to US$.

Use the multiplicative nature of the double number line to find pairs of values.

Example 4:

On this double number line, the 10 and 6 line up perfectly.



What other pairs of numbers will also line up in the same way?

V One aspect of variation is exploring the efficiency of different methods that can be used to solve a problem. In *Examples 4* and *5*, students may use different strategies to reach a solution.

It will be important for you to gather these solutions and, if necessary, suggest the strategy of using the functional multiplier of 0.6 if this is not suggested by students. For example, in *Example 5* the arrow is directly below the 2 on the top line and $2 \times 0.6 = 1.2$.

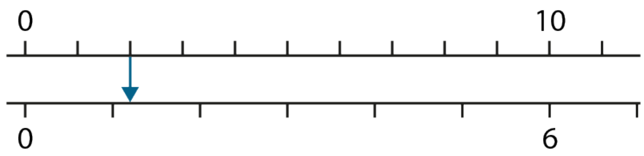
It will be important for students to realise that this functional multiplier of 0.6 will always change a number on the top line into the corresponding number on the bottom line.

Example 5:

g) Estimate the number that the arrow is pointing to.

h) Calculate the number that the arrow is pointing to.

i) Describe your calculation – what did you do?



PD As an alternative example, you could consider conversions between miles and kilometres. We know that 5 miles is 8 kilometres, and that, for *every* mile travelled, 1.6 kilometres is travelled.

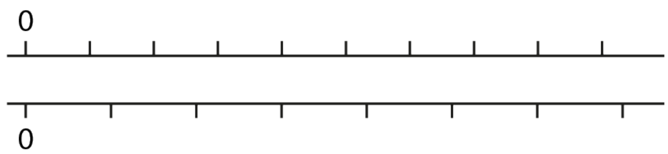
It is likely that, when converting between the two units, we will use different strategies according to the distances being worked on. Are there certain distances in which the multiplier of 1.6 is more likely to be used than scaling?

Use the double number line to represent multiplicative relationships, efficiently using both scalar and functional multipliers.

Example 6:

Each number line is marked in ones.

Fill in the labels on this double number line.



j) List some pairs of numbers that align perfectly. Could you list them all?

k) Which number is on the bottom line, directly under 2?

l) Which number is on the top line, directly above 2?

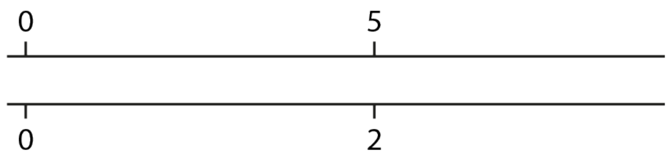
m) Which of these answers are you most sure of? How can you check?

R The double number line allows students to calculate accurately and estimate. *Examples 6* to *8* provide students with an opportunity to do both. Drawing students' attention to occasions when they use scaling along the line and when they use a multiplier between the lines is crucial. It is likely that students will prefer to work along the lines, and so may need greater experience with simple 'between the lines' (functional) multipliers to ensure that they are aware that this is also an option when calculating.

PD It is likely that students will need further experience of working with double number lines in which the scales are removed. Which numbers will you use in order to draw students' attention to the relationship between the lines? How will you ensure that students are aware of both multiplicative relationships?

Example 7:

Here is another double number line. Mark on the positions of some more numbers as accurately as you can.



n) List more pairs of numbers that align perfectly.

o) Which number is on the top line, directly above 3?

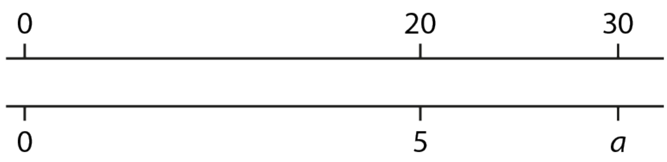
p) Which number is on the bottom line, directly below 3?

q) Which of these answers are you most sure of? How can you check?

PD Making connections between the double number line and other representations of multiplicative relationships can be challenging. You could ask students to consider the double number lines used in *Examples 7* and *8* and imagine that they are represented as a graph. Then, ask a question such as *'Which of the graphs will be the steepest? Make the connection between the graph and the double number line explicit to yourself.'*

Example 8:

Here is another double number line.



Steve says that, to find the value of a, he is going to calculate 5×1.5*.*

Mia says that she is going to find one quarter of 30.

Do you agree with Steve or Mia?

Explain how you know.

D In *Example 8* students will decide whether the functional or scalar multiplier is appropriate. The methods offered by Steve and Mia are both equally correct and may offer a catalyst for students to think about which multiplier they find easier to work with in each case. Students need to understand that they have a choice to make about which multiplier to use in a given context.

V You could ask students to replace one of the values given so that the functional multiplier is easier, and then to change it again so that the scalar multiplier would be the preference.

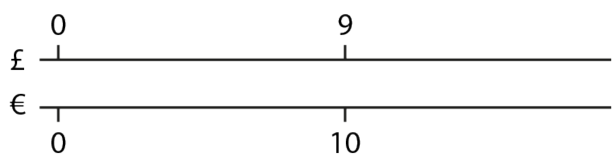
Use the multiplicative nature of the double number line to solve problems efficiently.

Example 9:

£9 is worth about €10.

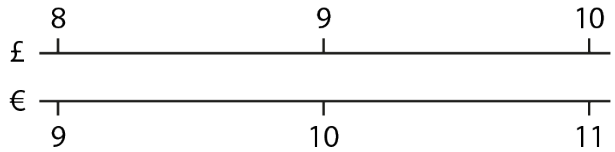
r) Which of the following double number lines shows this information correctly?

(i)

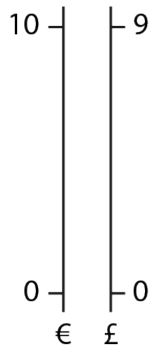


V Examples and non-examples of double number lines are used in *Example 9* as discussion points to ensure that students see the multiplicative, rather than additive, nature of double number lines.

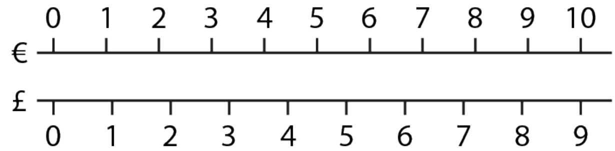
(ii)



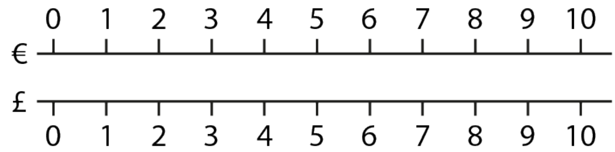
(iii)



(iv)



(v)



s) Use one of the correct double number lines to:

(i) Estimate how many pounds €7 is worth.

(ii) Calculate accurately how many pounds €7 is worth.

Example 10:

100 ml of orange drink is made with 80 ml water and 20 ml squash.

t) Show this information on a double number line.

u) Use your double number line to:

(i) Estimate the amount of water needed for 51 ml of squash.

(ii) Calculate accurately the amount of water needed for 51 ml of squash.

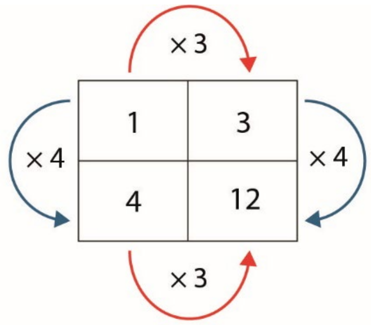
V *Examples 10* to *12* use a double number line in different contexts. Students should understand that the same representation can be used because the situations all share the same mathematical structure.

Example 11: *In a sale all items are reduced by 25%, so a shirt that usually costs £40 will be reduced by £10.* *v) Show this information on a double number line.* *w) Use your double number line to:* *(i) Estimate the amount to be taken off a jacket that usually costs £55.* *(ii) Calculate accurately the amount to be taken off a jacket that usually costs £55.*	**D** You could ask students to offer another context in which the same double number line could be used.
Example 12: *This rectangle is four times as tall as it is long.* *The rectangle can be made bigger or smaller on a screen, but its height is always four times the base.* *When the base of the rectangle is 13 cm long, the rectangle is 42 cm high.*  *x) Show this information on a double number line.* *y) Use your double number line to:* *(i) Estimate the length of the base when the height is 13 cm.* *(ii) Calculate accurately the length of the base when the height is 13 cm.*	**PD** Consider the 'ratio, proportion and rates of change' section of the Key Stage 3 mathematics programme of study, the chapter index of your Key Stage 3 textbook or your Key Stage 3 schemes of work. In which elements of these could a double number line be a productive representation?

3.1.2.2 Understand the language and notation of ratio and use a ratio table to represent a multiplicative relationship and connect to other known representations

Common difficulties and misconceptions

Vergnaud (1983) suggests that multiplication problems do not consist of a three-term relationship, rather a four-term relationship from which three relevant terms are extracted. In some simple situations, one of the terms may be 1. For example:



Using a ratio table can support students in identifying and arranging the three relevant terms from this four-term relationship, and in using this information to solve problems involving multiplicative structures.

It is important that students understand the structure that underpins the ratio table and that it is not viewed as a 'method'. Ratio tables are an abstraction of the double number line and, while a double number line drawn to scale has the advantage of allowing estimation, it is unlikely to be an efficient representation for solving a problem accurately. The ratio table allows multipliers to be quickly identified and a solution to be found to a multiplicative problem. Giving students opportunities to connect ratio tables with double numbers lines and the relevant symbolic notation is a key idea.

A challenge for students is to identify multiplicative situations, and the coherent use of a ratio table for these situations throughout the curriculum might support this. For example, the ratio table above could be used to identify equivalent fractions ($\frac{1}{4} = \frac{3}{12}$) or to share a quantity in a given ratio (A and B share money in the ratio 1:3. If A receives £4, how much does B receive). The use of this common representation supports student to see that the same structure underpins both contexts.

In multiplicative situations it is common for students to identify and work with the scalar multiplier rather than the functional multiplier (as discussed in this resource from the ICCAMS project), even when the functional multiplier is numerically easier. Using a ratio table makes both of these multipliers explicit and allows students to choose which to use.

What students need to understand	Guidance, discussion points and prompts
Identify the multipliers for information arranged in a ratio table *Example 1:* *Here is a section from a times-table square. A set of four numbers is highlighted.*	**R** The familiar times-table square should remind students that multiplicative relationships underpin this work. Moving the four connected points around the table square allows the multipliers to become less familiar (bringing in fractions for example) while reinforcing that the ratio table 'works' for any set of connected numbers.

×	1	2	3	4
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	4	8	12	16
5	5	10	15	20
6	6	12	18	24

In this diagram, there is a constant multiplier to move from the numbers in blue to those in red.

$$2 \times 2 = 4$$

$$3 \times 2 = 6$$

a) Find the constant multiplier to move from the blue to red.

b) Shift the group of four numbers to a different position on the table-square. Is there always a constant multiplier to move from left to right? What about from top to bottom?

L The table-square should remind students of the multiplicative nature of the relationships. This can also be supported by the language used, and reference to multipliers and multiplicative relationships can be helpful here.

D As students shift the set of four numbers around the grid, encourage them to separate the numbers to form the corners of a rectangle. For example:

×	1	2	3	4
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	4	8	12	16
5	5	10	15	20
6	6	12	18	24

Example 2:

Write down the multipliers and find the missing values in these ratio tables.

a)

4	20
12	

b)

4	5
6	

c)

10	11
9	

D These examples draw attention to the multiplicative connections. In part a) both the multipliers are familiar integers, allowing students to easily notice $\times$ 5 and $\times$ 3. Part b) then draws on this and students might be asked to consider the connections between the two different ratio tables. To move from 4 to 5, students may suggest a strategy of $\times$ 5 then $\div$ 4. It is important to encourage students to shift from this two-step process and work with the single operation of $\times \frac{5}{4}$.

In moving from 4 to 6, students might describe the multiplier as *'add half of 4'*. While correct, it is important that they are also able to see the multiplicative connection of $\times \frac{3}{2}$.

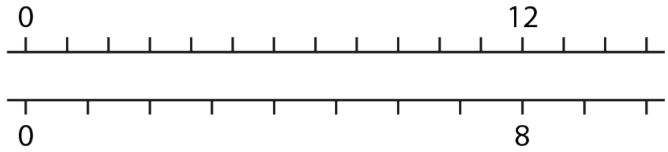
Part c) then moves away from familiar multipliers and includes a move from 10 to 9. Here, multiplication makes smaller, which some students may find challenging.

PD Consider the different ways that students might write the multipliers. For example, in part b) they might write $\times$ 1.25, $\times \frac{5}{4}$ or $\times 1\frac{1}{4}$.

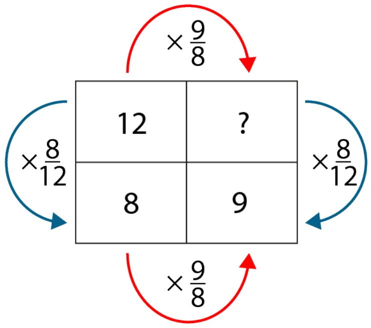
<table>
<tr>
<td></td>
<td>While all are correct, it might be argued that $\times\ \frac{5}{4}$ makes the connection more transparent as both the 5 and 4 are visible in the multiplier. To what extent do you agree with this? What are the benefits of the decimal and mixed representation?</td>
</tr>
<tr>
<td>
Understand ratio tables as an abstraction of the double number line.

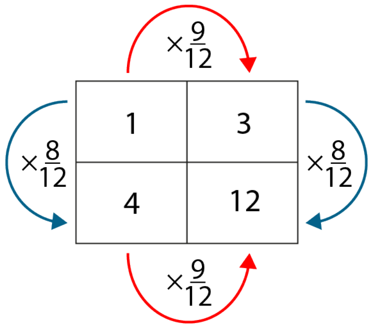
Example 3:

Ali buys eight identical packets of cakes for a birthday party. The total cost is £12.



a) Use the double number line to estimate the cost of 9 packets of cakes.
b) Which of these ratio tables could be used to calculate the cost of 9 packets of cakes?



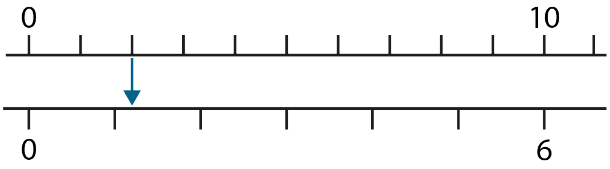


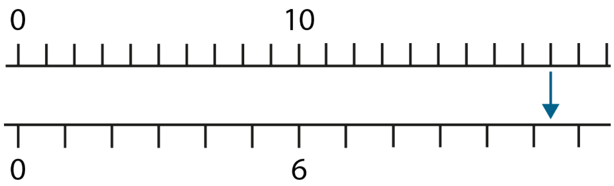
</td>
<td>
This context is taken from Example 2 of the exemplification for 3.1.2.1 Use a double number line to represent a multiplicative relationship and connect to other known representations.

Revisiting this task should support students in working between the different representations and making connections. This will help them identify the underpinning mathematical structure rather than see the ratio table as a ‘method’.
</td>
</tr>
</table>

Example 4:

Look at these double number lines.





For each of the double number lines:

a) Estimate the number that the arrow is pointing to.

b) Construct a ratio table and use it to calculate the number that the arrow is pointing to.

Example 4 also draws on the exemplification for *3.1.2.1 Use a double number line to represent a multiplicative relationship and connect to other known representations*.

V The two double number lines bring different challenges for students. In rewriting the first as a ratio table, students may find placement of the 2 challenging and the unknown value may be represented in an unfamiliar position. For example:

2	10
	6

This offers an opportunity to explore what it is possible to change within the structure underpinning the ratio table. For example, is this a valid representation for the double number line?

10	2
6	

The second double number line offered here is in a more familiar arrangement, but the multiplier is less familiar and may be perceived to be more challenging to work with.

Identify situations in which the ratio table is a suitable model.

Example 5:

1. This ratio table is filled in correctly.

30	24
100	80

Which of the following situations could be calculated using the ratio table?

a) The full price of a shirt is £30 but it is reduced by 20%. What is the new price?

V *Example 5* offers some contexts in which a ratio table can be used, and some examples of 'what it's not', with some situations that are not proportional and so cannot be represented in a ratio table.

PD The coherent use of a common representation such as a ratio table can support students in making connections between apparently different topics.

Consider the different contexts in which ratio tables might be used (*Example 5* includes similar shapes, percentage changes and sharing in a ratio) and whether the use of a ratio table in these contexts would

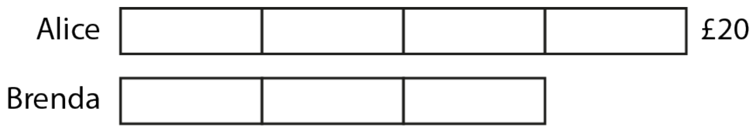
<table>
<tr>
<td>
b) Steve is 30 years of age and Paul is 24. When Steve is 100, how old will Paul be?
c) A rectangle is 30cm long and 1m high. How long is a similar rectangle that is 80cm high?
2. This ratio table is filled in correctly.
<table>
<tr><td>5</td><td>6</td></tr>
<tr><td>100</td><td>120</td></tr>
</table>
Which of the following situations could be calculated using the ratio table?
a) The fifth term of the linear sequence 10n + 50 is 100. What is the sixth term?
b) A can of paint is on offer at 20% extra free. This new can contains 6 litres of paint. How much paint does the standard can contain?
c) Kay and Grace share some money in the ratio 5:6. If Kay gets £100, how much does Grace get?
</td>
<td>support students in your school. To be effective, this representation should be consistent across the department (and the school). What are the barriers to this?</td>
</tr>
</table>

3.1.4.3 Be able to determine one part, given the other part and the ratio

Common difficulties and misconceptions

When solving problems involving unequal sharing, students may view a problem as a combination of multiplicative and additive processes.

R Consider the question *'Alice and Brenda share some money in the ratio* 4 : 3. *Alice has £20. How much does Brenda have?'* It is common for students to approach this problem by working multiplicatively: halve Alice's £20 to give £10, halve this again to give £5, and then additively combine these results to calculate that Brenda has £15. The use of a bar model representation in this situation can reinforce this combination of additive and multiplicative thinking:



However, the bar model, and other representations, can also be used to develop students' awareness of the multiplicative structure that underpins this problem, illustrating that Brenda's share of the money is $\frac{3}{4}$ of Alice's share in a way that is less apparent than when represented as the ratio 4 : 3.

The multiplicative relationship becomes more important as students work with relationships involving less 'friendly' numbers, where informal combinations of addition and multiplication can become unwieldy.

Students may assume that it is necessary to know (or to calculate) the total quantity being shared as a first step, and so find problems difficult to access.

R The use of different representations to make the relationship between the ratio and the two 'shared' parts can give students an opportunity to use more intuitive informal strategies.

What students need to understand

Interpret mathematically a situation involving unequal sharing, using correct notation and relevant diagrams.

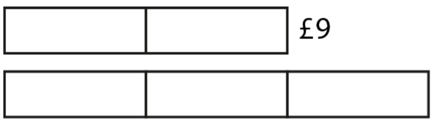
Example 1:

Oscar and Pietro share some money in the ratio 2 : 3.

Oscar's share is £9.

Explain how each of these diagrams can help to calculate Pietro's share of the money.

a)

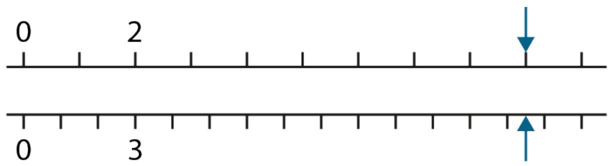


Guidance, discussion points and prompts

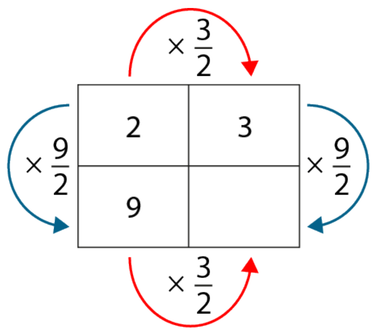
R It is important for students to be able to work between representations. Offering a situation, and asking students to represent it in more than one way, can be productive in making connections between those representations and the mathematical structures they represent.

It is also important to recognise the limitations of different representations. For example, if a question describes some money shared in the ratio 2 : 3 : 4, then a double number line or ratio table may not be as useful as a bar model representation, since the bar model can more obviously be adapted and interpreted to include the three-way split.

b)



c)



PD After completing *Example 1*, you could ask students to draw similar representations for a different situation, say, a ratio of 2 : 5.

Reflect on the representations that you habitually use when teaching students how to find one part, given the other part and a ratio. Why do you use those representations? What are the benefits? What are the drawbacks? Reflect on the representations offered in *Example 1*. What are the benefits of each? What are the drawbacks of each?

Example 2:

Mark and Ahmed share some sweets in the ratio 1 : 3.

Ahmed has eight more sweets than Mark.

How many does Mark have?

R
PD How could a bar model be used to solve this problem? This type of question is encountered in Key Stage 2 and is a good example of when a bar model is particularly efficient because there are additive structures as well as multiplicative structures in use. Useful activities and questions are ones that help students to recognise how the underlying structure of a problem can influence which type of representations are most appropriate.

Recognise the multiplicative relationship between *a* and *b* in the ratio *a* : *b*.

Example 3:

Richard and Sarah share some money in the ratio 2 : 5.

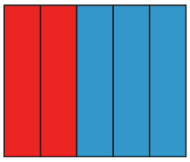
Louis says: 'For every two pounds that Richard has, Sarah has five pounds.'

Matilda says: 'Sarah always has two and a half times as much as Richard.'

List some possible amounts that Richard and Sarah could have and explain how they show that both Louis and Matilda are correct.

L The use of 'for every', as in *Example 3*, will link to other elements of multiplicative reasoning. For example, if sweets are shared between Alice and Brenda in the ratio 2 : 1, students should be able to read this as 'Alice has two sweets *for every* one sweet that Brenda has'.

Students' attention should be drawn to the two different multiplicative relationships used in *Example 3*. For any quantities shared in the ratio 2 : 5, one will always be 2.5 times greater than the other (this is often described as the functional multiplier). As students list pairs of quantities that fit this, they will use a second multiplicative relationship to scale up each of the quantities. For example, 2 : 5 can be scaled to 8 : 20 by multiplying by four (this is called the scalar multiplier).

	Students should understand that the functional relationship – the relationship between the two and the five – is constant and exists in all ratios. **R/PD** Consider this diagram:  What fractions do you see here? Many students will see only $\frac{2}{5}$ and $\frac{3}{5}$, but by shifting attention so that we see the red as the whole, we can also see that the blue is three halves of the red; and then shifting the blue to be the whole, we can see that the red is two-thirds of the blue. Representing the same relationship using ratio notation, 2 : 3, we can see that this notation contains all of these relationships. Two-fifths and three-fifths are evident, but also two-thirds and three halves, depending on which part of the diagram we view as the 'whole'.
Example 4: *Erin and Laura share some money.* *Laura has £3 more than Erin.* *Emily says: 'Erin and Laura have shared their money in the ratio 2 : 5.'* *Explain why Emily might think this and why she might not be right.*	**V** *Example 4* offers an example of *'what it's not'*. A common misconception for students working with ratio is to use additive rather than multiplicative relationships, assuming that a ratio of 2 : 5 means that Laura will always have £3 more than Erin. This question gives students a chance to explore and reason about why this cannot be the case. Students could list other ratios that would also give a difference of £3.
Example 5: *Sam and Cassidy share some money.* *James explains: 'Sam always has four times as much as Cassidy.'* *In what ratio have Sam and Cassidy shared the money? Explain how you know.*	**L** *Example 5* builds on *Example 4* and brings in another aspect of language, saying that Sam's share is *four times as much* (or four times greater) than Cassidy's. The use of this language, associated with scaling and enlargement, is designed to encourage students to think of the multiplication in another way, not just as repeated addition.

Example 6:

The ratio between the base length and the height of a rectangle is 3 : 1.

a) Describe the rectangle – is it short and fat or tall and thin?

b) The ratio changes to 3 : 2 – how has the shape of the rectangle changed?

c) The ratio changes again to 3 : 3 – how has the shape of the rectangle changed?

d) What is the ratio when the height is double the length of the base?

e) What is the ratio when the height is ten times the length of the base?

f) What is the ratio when the height is two and a half times the length of the base?

V In *Example 6*, students are asked to use the ratio 3 : 1 to imagine the shape of a rectangle, then imagine how that rectangle changes as the ratio changes – keeping the base length constant while varying the height.

Linger on the visualisation element of this task, only drawing the rectangle once students have had time to work on and understand the context (if at all).

Parts d), e) and f) then continue to use the geometrical context to focus attention on the multiplicative relationship between the base and the height of the rectangle.

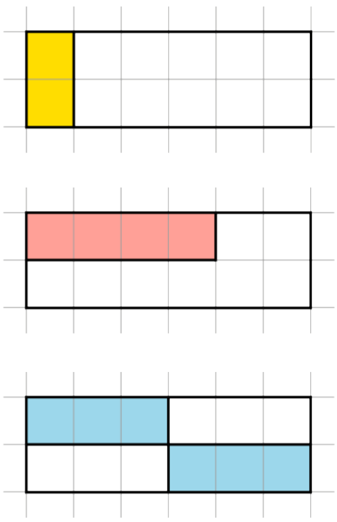
A geometric context can be useful when exploring the impact of changing a variable, as it offers a continuous, sliding representation for students to work with. It is worth noting that understanding the multiplicative relationship between the lengths of sides of a triangle is at the heart of GCSE trigonometry and so this is a key understanding for students to develop.

Find missing parts when a quantity is divided into more than two unequal parts.

Example 7:

All of these rectangles are to be coloured yellow : red : blue in the ratio 1 : 2 : 3.

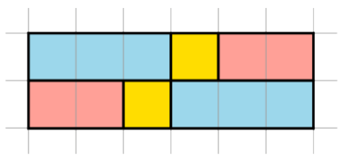
a) Complete the shading on each rectangle.



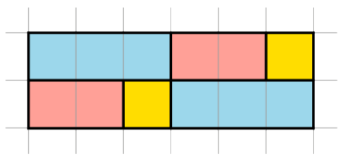
V In *Example 7* students will need to be aware that the three rectangles show the same relationship between the yellow : red : blue, although the parts might look different.

A prompt such as *'What's the same and what's different about your answers?'* might be a useful way to initiate students' reflections.

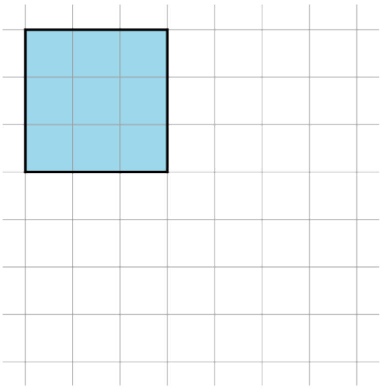
It should also be noted that the third rectangle in part a) can be considered as two smaller rectangles that are to be shaded in the same ratio. Shading that rectangle in a way that exploits this, for example:

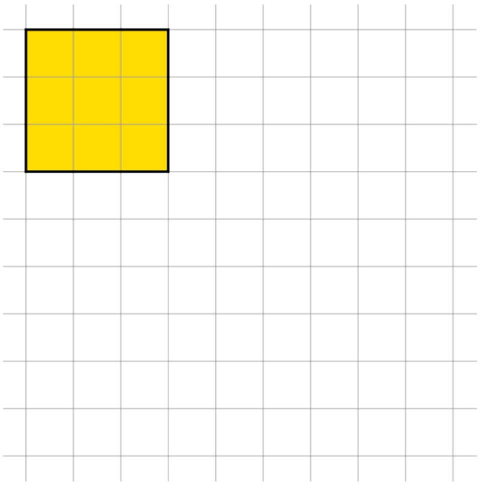


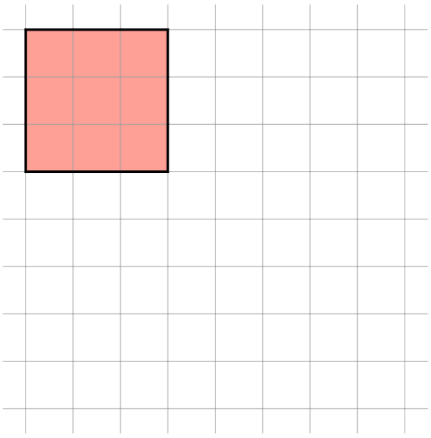
or:



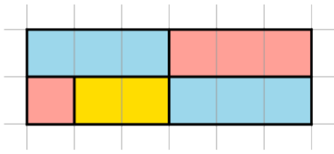
b) Draw the rest of each rectangle, then shade it in the same ratio as above.







might suggest a different understanding than shading the rectangle like this:



This may prove to be an interesting discussion point with students.

D To encourage students to consider the nature of the multiplicative relationships involved, you might like to ask students to consider what the ratio of yellow : red : blue would be if all of the rectangles drawn for part a) were combined.

V In part b) the size of the known section is constant and students have to use this to complete the other two sections, while still maintaining the rectangle. The intention here is that students become aware of the multiplicative nature of the relationships, which might be prompted further using questions such as *'What fraction of the blue area should the yellow area be?'*

Example 8: *Yasmine, Rachel and Brenda share some money in the ratio 1 : 2 : 3.* *a) How much do Yasmine and Brenda each have if Rachel has £9?* *b) Brenda always has three times as much money as Yasmine.* *(i) Write a sentence like this about the money that Rachel and Yasmine always have.* *(ii) Write a sentence like this about the money that Rachel and Brenda always have.*	**R** The images from *Example 7* can be used to support *Example 8* since both examples use the same mathematical structure. You might like to ask students to reflect on whether they preferred to work on the task represented pictorially or whether they preferred the contextualised representation. **L** In part b), writing the required sentences helps students to understand that even with a three-way split, the relationship between the parts is multiplicative.

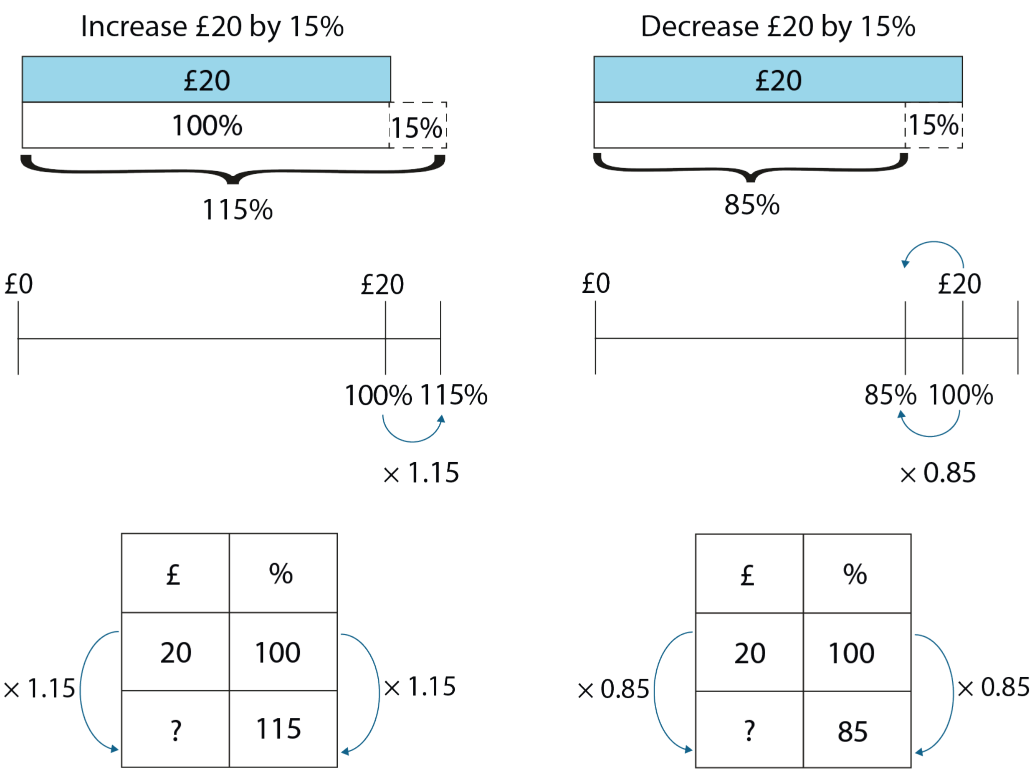
3.1.5.3 Calculate percentage changes (increases and decreases)

Common difficulties and misconceptions

Students should be confident with using informal additive methods to increase or decrease an amount using a percentage. While it is important that students can work flexibly with percentages, it is also important for efficiency and depth of understanding that they know there is a single multiplier linked to any percentage change and recognise this as an example of a multiplicative relationship.

Some students have difficulties using the additive method as they fail to find the final amount by adding or subtracting the increase/decrease to/from the original amount (see *Example 1*). Some students have difficulties with identifying the multiplier for single digit percentages such as 5% (see *Example 4*).

R Bar models, double number lines and ratio tables are all powerful representations that can help students work 'beyond 100%' and identify the both the whole, and the multiplier linked to the percentage.



PD Do you use these representations to support and develop students multiplicative reasoning?

What students need to understand	Guidance, discussion points and prompts
Find a percentage increase or decrease using an additive method. *Example 1:* *Increase:* *a) £30 by 50%* *b) £50 by 30%*	**V** The questions in *Example 1* have been designed to help students notice that although the percentage of the amount is the same value (£15), the outcome is different.

Example 2:

Mary is decreasing 40 kg by 20%. She says that the answer is 8 kg because 20% of 40 kg is 8 kg.

Do you agree? Explain your answer.

The 'what's it's not' question in *Example 2* supports students' awareness of the need to decrease the original amount by the proportion and not just find the proportion.

PD How would you convince yourself and your students that finding *x*% of £*y* gives the same result as finding *y*% of £*x*? How would you convince yourself and your students that increasing £*y* by *x*% does **not** give the same result as increasing £*x* by y%?

PD Here students are introduced to percentage changes using an additive approach before the use of a single multiplier to calculate the change. Consider the benefits and disadvantages of this route to understanding percentage change.

Use multipliers to calculate percentage increase or decrease.

Example 3:

Tick the correct statements:

	Calculation	**Tick**
Increase £20 by 35%	20×1.35	
Decrease £20 by 35%	20×0.35	
Increase £35 by 20%	35×0.2	
Decrease £35 by 20%	35×0.8	
Increase £45 by 1%	45×1.1	
Decrease £35 by 1%	45×0.99	

V The questions in *Example 3* are intended to help students recognise that there is a single multiplier linked to any percentage change. Multiplicative reasoning can increase a quantity by a percentage more efficiently than the additive methods in *Examples 1* and *2*.

PD Students need to be confident and fluent with recognising that a single multiplier exists linked to any percentage. How might you assess this before starting *Example 3*?

Example 4:

a) Sandra tries to increase £40 by 5% by working out 40×1.5. Explain why she is incorrect.

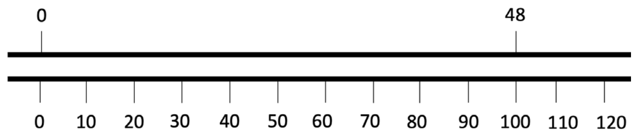
b) Nicola tries to decrease £40 by 5% by working out 40×0.05. Explain why she is incorrect.

Example 4 provides opportunities to assess students' understanding of the single multiplier linked to any percentage and to discuss some common misconceptions.

PD Do you value the importance of asking students to discuss and explain why statements are incorrect, or asking them to improve upon given answers? How do you become aware of common misconceptions such as that addressed here?

Example 5:

A chocolate bar that usually weighs 48 g is on special offer and now includes '15% extra free'.



a) Mark the new weight on the double number line.

b) Use the double number line to estimate the new weight.

The same percentage increase can be calculated using this ratio table.

100	115
48	

c) Mark the multipliers on the ratio table.

d) Use the ratio table to calculate the new weight.

R *Example 5* provides an opportunity to connect ratio tables and double number lines to percentage changes.

Your students may be familiar with ratio tables as representations of multiplicative relationships. Their use here will support students in connecting percentage changes with other situations that share the same mathematical structure, even though they initially look different.

PD Part b) asks students to use the double number line to estimate the new weight, but part d) asks them to use the ratio table to calculate. It could be argued that the double number line allows an insight into the structure while the ratio table (which is an abstraction of the double number line, with the same information represented but without the scale) allows easy identification of the multiplier and hence calculation of the solution. To what extent do you agree with this perspective?

Solve familiar and unfamiliar problems, including real-life applications.

Example 6:

a) A ball is dropped from a height of 200 m. After each bounce, the ball reaches 10% of its previous height. How high does it reach after the third bounce?

b) A shop is offering 25% off items in a summer sale. On the final day of the sale, all prices are reduced by 10%. Debbie buys a shirt on the final day. Work out the overall percentage reduction on the price of the shirt.

V *Example 6* provide two opportunities for students to apply their understanding to solve unfamiliar problems. Part b) does not include a quantity, to focus students' awareness on the effect of combining more than one multiplier rather than calculating an answer.

Example 7:

Complete the statement in several different ways:

a) Increase £ $\boxed{}$ *by* $\boxed{}$ *% = £80*

b) Decrease £ $\boxed{}$ *by* $\boxed{}$ *% = £80*

D *Example 7* offers two empty box problems that have more than one solution. Questions like these can encourage students to consider the overall structure of increasing and decreasing a quantity by a percentage.

Students might need prompted to consider percentages greater than 100%, but this can also offer a useful insight into their understanding. Asking students to

	communicate the process they went through to find a solution will also help refine their mathematical thinking.
Example 8: *Is the following statement always true, sometimes true or never true? Explain your thinking* *'Increasing a quantity by x% and then reducing the new quantity by x% will give an answer that is the same value as the original quantity.'*	**D** Statements such as the one in *Example 8* provide opportunities for students to 'play' and conjecture when increasing and decreasing quantities by a percentage and explore the structure to greater depth. Using a representation such as a double number line or ratio table can help make sense of this structure. **PD** How do you manage questions like *Examples 7* and *8* with your classes? How long would you give them before you intervene and support? What prompts could you give?

Weblinks

1. NCETM primary mastery professional development materials
   https://www.ncetm.org.uk/resources/50639
2. NCETM primary assessment materials
   https://www.ncetm.org.uk/resources/46689