9 Sequences, functions and graphs

Mastery Professional Development

9.2 Exploring non-linear sequences

Guidance document | Key Stage 4

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Click the heading to move to that page. Please note that these materials are principally for professional development purposes. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

Making connections

Building on the Key Stage 3 mastery professional development materials, the NCETM has identified a set of five 'mathematical themes' within Key Stage 4 mathematics that bring together a group of 'core concepts'.

The third of the Key Stage 4 themes (the ninth of the themes in the suite of Secondary Mastery Materials) is *Sequences, functions and graphs*, which covers the following interconnected core concepts:

9.1 Exploring linear equations and inequalities

9.2 Exploring non-linear sequences

9.3 Exploring quadratic equations, inequalities and graphs

9.4 Exploring functions

9.5 Exploring trigonometric functions

This guidance document breaks down core concept *9.2 Exploring non-linear sequences* into two statements of **knowledge, skills and understanding**:

9.2 Exploring non-linear sequences

- 9.2.1 Explore geometric sequences
- 9.2.2 Understand and work with quadratic sequences

Then, for each of these statements of knowledge, skills and understanding we offer a set of **key ideas** to help guide teacher planning:

- 9.2.1 Explore geometric sequences
  - 9.2.1.1 Recognise that the numbers that form a sequence are discrete values
  - 9.2.1.2 Appreciate that a sequence can change in a non-linear way
  - 9.2.1.3 Recognise and interpret geometric growth in a sequence
- 9.2.2 Understand and work with quadratic sequences
  - 9.2.2.1 Revisit the structures that underpin the nth term for linear sequences
  - 9.2.2.2 Recognise and interpret quadratic growth in a sequence
  - 9.2.2.3 Understand and use method(s) to express the nth term of a quadratic sequence

Overview

This core concept extends students' understanding of sequences to look more deeply at non-linear sequences. It sits within the wider theme of functions and graphs because sequences relate a domain member with a range member, but a key understanding for students to appreciate is the distinction between the discrete nature of sequences and the continuous nature of functions.

At Key Stage 4, the focus is on extending students' understanding of sequences to encompass more non-linear sequences. Students will have already had experience of some common non-linear sequences, such as square, cube and triangular numbers, so will have an awareness that sequences can grow by a different amount with each consecutive term. However, they will not necessarily have considered the features that distinguish different types of non-linear sequences. A key piece of new learning is the definition of a geometric sequence, where the term-to-term rule is multiplicative. Students should be able both to recognise geometric sequences and to articulate how they differ from arithmetic sequences, where the term-to-term rule is additive.

Students should also realise that not all non-linear sequences are formed geometrically, with a particular emphasis on quadratics. Students will build on their understanding of the nth term for linear sequences, established at Key Stage 3, and apply this to finding the nth term for quadratic sequences. There are many different methods which teachers may use to teach this, but the emphasis should be on revealing the underlying structures and ensuring students have an appreciation of how the expression for the nth term is generated. It may help students to recognise the arithmetic sequence that is embedded within the quadratic. The examples in exemplified key idea 9.2.2.3 offer a range of approaches to support this conceptual understanding, including pictorial representations and exploration of the common difference between the differences.

Students' wider understanding of functions is developed in this core concept. Sequences can provide a context in which to think about the way the value of a function's output changes when the value of the input is changed. This linking is especially important with non-linear functions because the change can be unexpected and dramatic. Students should then be guided from the specific case in sequences to a more general understanding of the link between pairs of values connected by a functional relationship.

Prior learning

Students will have explored patterns and sequences from their earliest experiences of mathematics at primary school. During Key Stage 3, the focus was on formalising this experience so that they were able to continue, generate and describe arithmetic sequences using both term-to-term and position-to-term rules. They might use the terms 'arithmetic' and 'linear' interchangeably, but should be aware that the term 'arithmetic' describes sequences where each term is generated by adding a constant value to the previous term. They should be familiar with the structure of such sequences and understand what is meant by, and be able to write, an expression for the nth term. Students may have connected their work on arithmetic progression sequences with their work on linear graphs, recognising that consecutive terms and consecutive y-coordinates can both be found by adding or subtracting a constant amount to the preceding value. They should have some understanding that an arithmetic sequence is discrete, whilst a linear graph is continuous, but this distinction should be reinforced throughout this theme.

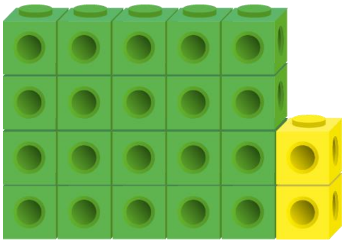
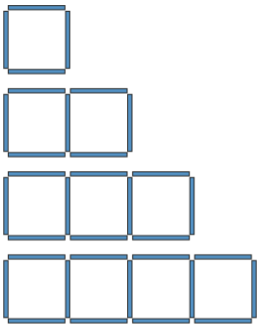
Students should be aware that sequences do not always grow in a linear fashion, including experience of common non-linear sequences such as square numbers, cube numbers and triangle numbers.

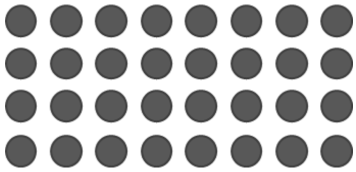
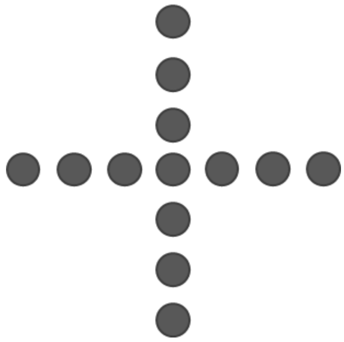
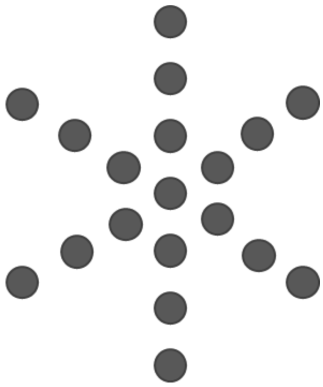
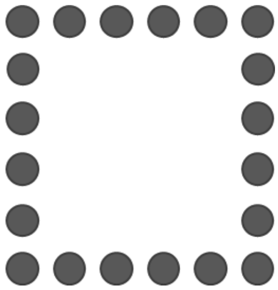
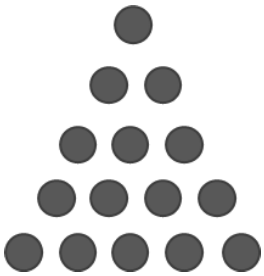
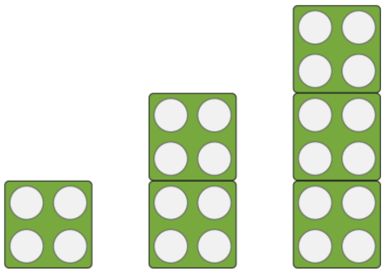
Prior work with quadratic expressions will likely have included expanding and simplifying pairs of brackets, so they will be familiar with the structure $ax^2 + bx + c$. Students are less likely to have met quadratic functions or be aware of the general shape that these functions will take when plotted on a Cartesian graph. Careful consideration is needed when thinking about the order in which quadratic graphs and quadratic sequences are taught at Key Stage 4. Links will need to be made both across to other contexts for quadratics such as graphs, and also to the structure of linear sequences so that connections can be made with finding the nth term.

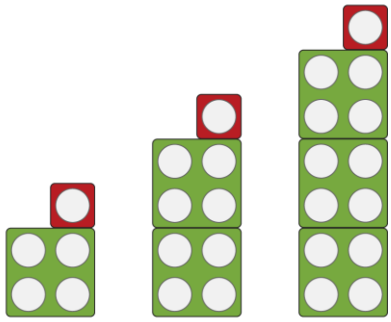
4.1 Sequences from the Key Stage 3 PD materials explores the prior knowledge required in more depth.

Checking prior learning

The following activities from the NCETM Secondary Assessment materials, Checkpoints and/or Key Stage 3 PD materials offer a sample of useful ideas for assessment, which you can use in your classes to check understanding of prior learning.

Reference	Activity
Secondary Assessment materials, page 19	Which is bigger: $2n$ or n^2? Explain your answer.
Secondary Assessment materials, page 22	The picture shows the 5th term of a pattern made with cubes to represent the sequence $4n + 2$.  • What in the picture shows that it's the 5th term? • What in the picture shows that $4n$ is part of the rule for the sequence? • What in the picture shows that $+2$ is part of the rule for the sequence?
Key Stage 3 PD materials document '*4.1 Sequences*', Key idea 4.1.2.2, Example 1	The following sequence of growing shapes is made up of sticks.  It generates the following sequence of numbers (the number of sticks): 4, 7, 10, 13… Draw the 17th diagram. Think about the structure of the diagram as you draw it. How would you work out the number of sticks in the 17th diagram without counting them one by one? What calculation would you perform?

<table>
<tr>
<td>Adapted from Checkpoints ‘Sequences’, Checkpoint 5 ‘Snapshots’</td>
<td>
Briefly reveal each pattern to students, asking the following questions:
How many dots are in each pattern?
How do you know? Describe how you saw the pattern.
A

B

C

D

E

If each pattern was part of a different sequence, what might the next pattern be for each? What might the previous pattern have been?
</td>
</tr>
<tr>
<td>Checkpoints ‘Sequences’, Checkpoint 13 ‘Tiles and holes’</td>
<td>
A sequence of patterns is made from green tiles, each with four holes.
a) How many green tiles are in the 3rd pattern? The 6th pattern? The 10th pattern?
b) How many holes are there in the 3rd pattern? The 6th pattern? The 10th pattern?

</td>
</tr>
</table>

	c) Describe in words the relationship between the number of tiles and the number of holes. d) How does your answer to part c change if the sequence looks like the second sequence, with green and red tiles, instead? e) If a pattern has 217 holes, which of the two sequences is it from? Describe what the pattern looks like. 

Key vocabulary

Key terms used in Key Stage 3 materials

- arithmetic sequence
- geometric sequence
- nth term of a sequence
- sequence
- term

The NCETM's mathematics glossary for teachers in Key Stages 1 to 3 can be found here.

Key terms introduced in the Key Stage 4 materials

Term	Explanation
first/second difference	First difference, in the context of sequences, refers to the difference between adjacent terms in a sequence. If the first difference is constant, then the sequence is arithmetic. Second difference refers to the difference between adjacent first differences. If the second difference is constant (i.e. the sequence of first differences is an arithmetic sequence), then the sequence is quadratic.
quadratic sequence	A quadratic sequence is a pattern of numbers characterised by the difference between each term increasing or decreasing at a constant rate (i.e. the second difference is constant). The expression for the nth term of a quadratic sequence will always contain an n^2 term and so can be generalised as $an^2 + bn + c$.

Knowledge, skills and understanding

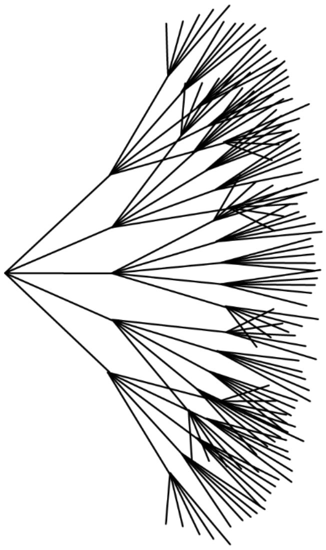
Key ideas

In the following list of the key ideas for this core concept, selected key ideas are marked with a . These key ideas are expanded and exemplified in the next section – click the symbol to be taken direct to the relevant exemplifications. Within these exemplifications, we explain some of the common difficulties and misconceptions, provide examples of possible pupil tasks and teaching approaches and offer prompts to support professional development and collaborative planning.

9.2.1 Explore geometric sequences

The work in this core concept extends students' understanding of geometric sequences. At Key Stage 3, students were expected to have some experience of sequences that did not grow arithmetically, and this exploration continues into Key Stage 4. They should already have had some opportunities to explore the structure of some special number sequences such as square numbers, cube numbers and triangle numbers. At Key Stage 4, other sequences such as the Fibonacci sequence should be included in this list.

There is an opportunity to start to explore exponential sequences, seeding later learning on exponential functions in '*9.4 Exploring functions*'. The continually changing rate of growth is a feature of exponentiation that students may need to grapple with. It can be tempting for students to mentally categorise all non-linear relationships together and not appreciate the features that set them apart. For example, in polynomial sequences there is always a point at which the difference between terms will become constant. With quadratic sequences this is the second difference, with cubics the third, and so on. However, this is not true of exponential sequences or functions: the rate of change is never constant, no matter how many times it is iterated. Students may need to begin by considering exponentiation as repeated multiplication, in a similar way that multiplication can be thought of as repeated addition. Pictorial representations of exponential relationships can be a powerful tool in developing this understanding. For example, the image to the right is taken from '*7.2 Using structure to transform and evaluate expressions*', and gives some insight into the rate of growth in an exponential sequence.



9.2.1.1 Recognise that the numbers that form a sequence are discrete values

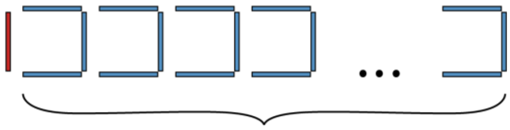
9.2.1.2 Appreciate that a sequence can change in a non-linear way

9.2.1.3 Recognise and interpret geometric growth in a sequence

9.2.2 Understand and work with quadratic sequences

The progression in understanding sequences may develop through noticing a pattern, to finding other members which follow the pattern and then on to generalising a rule. However, following a pattern alone is less productive than understanding the structure of which that pattern is a consequence, and representations such as counters and matchsticks can help make this structure more apparent to students.

Examples of sequences requiring students to determine the next term, or the next +1 term are useful if they draw attention to the structure of the sequence.



17 groups of 3 sticks

In the core concept document '*4.1 Sequences'* from the Key Stage 3 PD materials, the example above was used to prompt students to consider the pattern, encouraging them to pay attention to the role of the '17' in their explanation of the sequence. In using 17 as a variable and considering how that number would change as more groups of 3 sticks were added, students should have come to appreciate the structure of the pattern and how it relates to the algebraic expression for the nth term. They should understand that the coefficient, in this case '3', is related to the 'difference' between terms, and that the variable is related to the position of the term within the sequence (and that the position corresponds to exactly one term). The idea of difference, and particularly constant difference, is key to understanding the structure of quadratic sequences and how their growth is distinct from linear sequences, a new understanding for Key Stage 4.

Finding the nth term of a quadratic sequence is often a procedure that students learn as a disconnected method. As with linear sequences at Key Stage 3, the representations within this core concept are used to access some of the structure that underpins this method. For instance, *Example 4* of key idea 9.2.2.3 uses counters to look more deeply at the second difference being constant, rather than the first.

Working with sequences also gives a context to explore a key difference between quadratic sequences and quadratic functions: that the function can be understood as an object, whereas the sequence remains a series of ordered pairs. By using a common language and paying attention to the similarities and differences, students should develop a sense of when relations can be described as a function, and when they cannot. This idea is explored further in '*9.3 Exploring quadratic equations, inequalities and graphs'* and '*9.4 Exploring functions'*.

9.2.2.1 Revisit the structures that underpin the nth term for linear sequences

9.2.2.2 Recognise and interpret quadratic growth in a sequence

9.2.2.3 Understand and use method(s) to express the nth term of a quadratic sequence

Exemplified key ideas

In this section, we exemplify the common difficulties and misconceptions that students might have and include elements of what teaching for mastery may look like. We provide examples of possible student tasks and teaching approaches (in italics in the left column), together with ideas and prompts to support professional development and collaborative planning (in the right column).

The thinking behind each example is made explicit, with particular attention drawn to:

Deepening	How this example might be used for **deepening** all students' understanding of the structure of the mathematics.
Language	Suggestions for how considered use of **language** can help students to understand the structure of the mathematics.
Representations	Suggestions for key **representation(s)** that support students in developing conceptual understanding as well as procedural fluency.
Variation	How **variation** in an example draws students' attention to the key ideas, helping them to appreciate the important mathematical structures and relationships.

In addition, questions and prompts that may be used to support a professional development session are included for some examples within each exemplified key idea.

These are indicated by this symbol.

9.2.1.3 Recognise and interpret geometric growth in a sequence

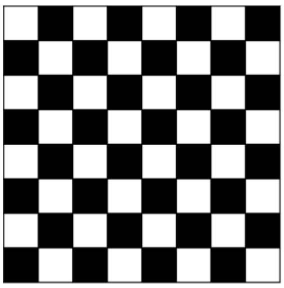
Common difficulties and misconceptions

When students start to explore beyond linear sequences, it is common for them to mentally group all non-linear sequences together, and not appreciate the fact that there are different types of non-linear growth. It is important that students gain an appreciation of the features that distinguish geometric sequences from other types of sequences, and that there are opportunities to compare different types of non-linear sequences. Drawing attention to which properties are the same and which are different (for example, between the powers of 2 and the square numbers) can help students to develop more accurate definitions of each type of sequence.

Conceptualising geometric growth is challenging and the increasingly large (or increasingly small) differences between each term within a sequence can feel counter-intuitive. It can be helpful to start by considering such sequences as 'repeated multiplications", in the same way that a linear sequence can be thought of as 'repeated additions'. This can also help to clarify the difference between, say, geometric and quadratic sequences: in a quadratic sequence, the difference between terms does increase by 2 each time, but this 2 is constant and so it is still a sequence formed on the basis of repeated addition rather than repeated multiplication. The features of geometric growth established here are further consolidated for students in exemplified key idea 9.2.2.3, which explores quadratic equations.

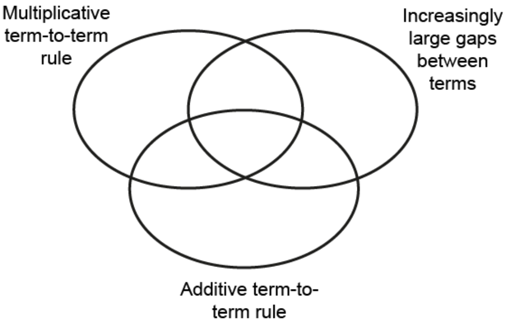
It is also important that students do not solely focus on geometric sequences where the terms increase in size each time, and appreciate that, when the multiplier is between 0 and 1, the gap between terms will decrease each time. Take care to ensure students experience a range of different examples to prevent misconceptions forming.

Repeated multiplication is one structure inherent in geometric growth, and it is particularly helpful when looking in discrete contexts such as the sequences explored in this key idea. However, it is important to note that, when students start to look at geometric growth in continuous contexts such as graphs, the repeated multiplication structure can be seen as unhelpfully ‘chunky’ and segmented. This is addressed further in core concept document ‘*9.4 Exploring functions’*.

Students need to	Guidance, discussion points and prompts
Begin to appreciate the increasing/decreasing difference between terms in a geometric sequence *Example 1:* *You are offered two different options:* • *Option A: £10 000 every day for one month* • *Option B: £1 on the first day of the month, £2 on the second day, £4 on the third, £8 on the fourth and so on to the end of the month.* *a) Which option would you prefer to have?* *b) Explain why.* *In the year this is offered, February has 28 days and March has 31 days.* *c) How much more would you receive on the last day if Option B was offered in March compared with February?*	This opening example is a classic ‘would you rather?’ comparison that draws attention to the rate at which a geometric sequence grows. It is likely to be counter-intuitive for students that, in as little as a month, a doubling sequence starting on £1 could possibly result in higher quantities than a constant supply of £10 000. The use of money and calendar months is intended to lend some familiarity to a conceptually difficult piece of mathematical thinking. A physical **representation** of a calendar may be useful in supporting explanations. Pay careful attention to the **language** used to explain the difference between the two options. An efficient way to calculate the total for the month for Option A is to multiply £10 000 by the number of days. Students should not get confused between this multiplication and the repeated multiplication between terms for Option B. Make sure students are clear that the difference between each day/term is constant for Option A, but that it changes each time for Option B. Part c, which narrows the focus to just three days, might be helpful in reinforcing this point.
Example 2: *Grains of rice are placed on each square of the chessboard. The number of grains of rice in each square is double that of the previous square.*  *a) If there are 128 grains of rice in the last square of the first row, how many grains of rice were there in the first square?* *b) How many grains of rice will there be by the end of the second row?*	As with *Example 1,* the chessboard problem is an oft-used context for exemplifying the surprisingly rapid nature of geometric growth. You may find it helpful to use dots or markings as **representations** of the grains of rice, if only to demonstrate how quickly the number of grains becomes too hard to draw within one square. With this in mind, the questions in this example do not require students to focus immediately upon the unimaginably large number of grains of rice on the 64^{th} square. Rather, they are designed to draw attention to more familiar large numbers (such as one million) and consider their position on the board relative to other values. Compare this example and *Example 3* of exemplified key idea 9.4.1.3, with the view to **deepening** students’ appreciation of growth in a discrete context such as a sequence when compared to growth in a continuous context. Graphical representations could be used to demonstrate that, in the case of the chessboard, there is no ‘4.5^{th}’ square, where 12 grains would be found; however, in the case of the algae, there is a measurement of algae growth that could be taken on the 4.5^{th} day. (Note: while the growth of algae on the pond’s surface may

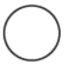
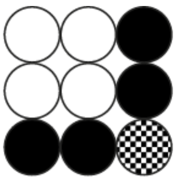
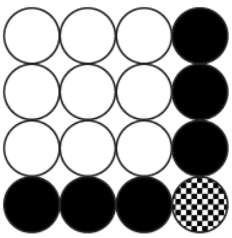
c) In which row will the number of grains of rice exceed 1 million? *d) In which row will the number of grains of rice exceed 10 million?* *e) How about 100 million? 1 000 million?*	appear continuous, it could be argued that there is still a discrete number of organisms.)
Example 3: *Bela draws a line. She says, 'I'm going to draw a set of lines, where each line is 10% longer than the last one.'* *Bela thinks that the 10th line she draws will be double the length of her original, because 10 lots of 10% is 100%.* *a) Draw a set of lines that increase in length by 10% each time.* *b) Explain why Bela's thinking is wrong.* *Bela then decides to draw a set of lines where each line is 10% shorter than the last one. She says, 'I think I will be able to draw nine lines. By the time I get to the 10th line, there will be nothing to draw: 10 lots of 10% is 100% and so I'm taking away the length of the whole line.'* *c) Explain what Bela's 10th line will really look like.*	Compound interest can be challenging for students to fully comprehend, and so *Example 3* offers a tangible **representation** to help them to visualise this concept. The key learning to draw out is that a geometric sequence is not linear, so the increase in length for each line is itself increasingly big. Part c essentially offers the same misconception but instead the sequence of lines is reversed, so each subsequent line diminishes in length. It is intentional that the context remains the same throughout, rather than introducing a new scenario (as in *Example 4*, below). This **variation** draws attention to the features that are the same: that the value of 10% changes each time, and that both ascending and descending geometric sequences can continue infinitely. It also emphasises the features that are specific to geometric sequences where the terms have increasingly small differences. For example, that subsequent terms will tend towards, but not meet, 0. These features will be built on in later work on graphing exponential functions.
Example 4: *Sarah and Roz are sharing a cake. There is one slice left.* *Sarah eats half of the slice, leaving half for Roz. Roz then eats half of what is left, leaving half for Sarah. Sarah then halves what is left, and so on.* *Who will finish the cake? How do you know?*	It is important for students to understand that geometric sequences include sequences where the terms decrease in size, and so the difference between terms is increasingly small. *Example 4* offers a scenario that students may well have come across in real life, but its place in this set of examples is designed to connect explicitly with geometric growth and give support in **deepening** students' definitions. The question is impossible to answer, to give students the opportunity to reason about how, numerically, this sequence will continue infinitely (even if, physically, the crumbs of cake become too small to cut). Here, the multiplier between each term is one-half, but students should be encouraged to generalise about all sequences where the multiplier is between 0 and 1.

Understand that geometric growth can be thought of as repeated multiplication *Example 5:* *Find the missing terms in each sequence* *a) 1, 2, _, _, 16, _* *b) 1, _, _, 1 000, _* *c) 1, _, 36, _, _* *d) _, _, _, _, 256, 1 024*	*Example 5* draws students' attention to the multiplicative relationship between subsequent terms in a geometric sequence. The question prompt asks for missing terms, and so it will be important to supplement this activity with prompts to elicit the term-to-term rule being used each time. Students who have less experience of non-linear sequences may start by trying to find an additive relationship by subtracting terms and equally sub-dividing the difference between the given gaps. Teachers should consider how they might encourage students to consider a different relationship. For example, by considering the different calculations that can result in 2 from a starting point of 1. The **variation** in the four sequences used, with each starting from the same value and growing differently, should support such thinking.
Example 6: *The prefixes of metric units are 1 000 times larger than each other.* *One kilowatt is 1 000 times more than one watt.* *One megawatt is 1 000 times more than one kilowatt.* *One gigawatt is 1 000 times more than one megawatt.* *One terawatt is 1 000 times more than one gigawatt.* *a) How many more watts are there in a megawatt than in a kilowatt?* *b) How many more watts are there in a terawatt than in a gigawatt?*	*Example 6* uses the **language** of metric prefixes to draw attention to the changing differences between consecutive terms. Students might instinctively answer '1 000 more' and need prompting to remember that each consecutive unit is '1 000 **times** more'. The difference between the answers for part a and part b may surprise students, but will emphasise the key feature of ever-increasing growth in a geometric sequence. Students are asked for the difference between megawatts and kilowatts, then terawatts and gigawatts. It might be tempting to 'fill the gap' and ask for a comparison between gigawatts and megawatts. Do you feel there would be a benefit to making more comparisons? Why or why not?
Appreciate the difference between linear, non-linear and geometric sequences *Example 7:* *25 and 36 are two consecutive terms of a sequence.* *What do you expect the next term to be if the sequence is:* *a) quadratic* *b) linear* *c) exponential?*	The **variation** of *Example 7* keeps the two consecutive terms the same, so that students need to focus on the features of the different types of sequence. The intention is that students can articulate that the subsequent terms will be increasingly far apart for the two non-linear sequences, and that this growth will be increasingly large for exponential sequences. Quadratic sequences are explored in detail in exemplified key idea 9.2.2.3 later in this document. There is no specified order for working through the key ideas, so your scheme of learning may lend itself to exploring quadratic sequences before geometric sequences. Reflect on this example and consider how differently students might approach it if they already have experience of establishing rules for quadratic sequences.

Example 8: *Here are some sequences:* *1, 3, 6, 10, …* *1, 3, 9, 27, …* *1, 3, 5, 7, …* *a) Position the sequences within the Venn diagram below, according to their properties.*  *b) Write suitable sequences with a single step term-to-term rule in the remaining gaps in the Venn diagram. Are there any that are not possible? Why or why not?*	The **variation** of *Example 8* is such that all three of the given sequences start with the same two numbers (1, 3, …) to draw attention to how they each grow differently beyond that. In positioning the given sequences for part a, and then creating and positioning their own sequences for part b, students should think carefully about whether all of the intersections are possible to complete. A term-to-term rule with a single step, as specified here, cannot be both additive and multiplicative. The term 'additive' is used, rather than 'arithmetic', to encapsulate all sequences where terms are generated by addition, not just those with a constant first difference. A Venn diagram can be a useful tool for categorising properties, but it is important to anticipate potential over-generalisations and misconceptions. For example, students might think that the triangular numbers are the only example of an additive term-to-term rule with increasingly large gaps between terms when; in fact, any polynomial of order 2 or more could fulfil these criteria (see the guidance for *Example 13* from 9.2.2.3 below). Consider what prompts you might give to ensure students generate a broad range of appropriate sequences. For example: • *'Have you included any descending sequences?'* • *'Where would you position quadratic/cubic sequences?'* • *'Would all linear sequences go in the same place?'* • *'How would part b change if your term-to-term rules could have more than one step, such as **add 2 then multiply by 6**?'*
Example 9: *The rules for the* n*th terms of three different sequences are below.* *A:* $15n + 1$ *B:* $10n^2 + 1$ *C:* $5^n + 1$ *Decide which sequence grows the fastest and explain how you know.*	The final example in this sequence is the first to use the algebraic notation for nth term. At Key Stage 4, students must be able to find and work with the nth term for linear and quadratic sequences, but not geometric. The inclusion of $5^n + 1$ here is to help students to draw a distinction between those familiar sequences and sequences that grow exponentially. The focus is on **deepening** students' understanding, so the question asks generically about comparing the rate of growth rather than specifically to generate any values. Links can be made to work on exponentiation and understanding its structure of repeated multiplication from '*7.1 Using structure to calculate and estimate'*.

9.2.2.3 Understand and use method(s) to express the nth term of a quadratic sequence

Common difficulties and misconceptions
At Key Stage 3, students worked on the structures that underpin linear sequences and developed strategies to find the nth term. If this is well understood, it provides a strong foundation for working with quadratic sequences. The method for finding the nth term of a quadratic sequence is often taught as a procedure with little or no connection to the mathematical structures that underpin it. Without this insight, students are more likely to make errors when applying the method. These materials aim to give meaning to the structure by using familiar representations – in this case, counters arranged in arrays. If students are not familiar with this representation, consider whether other more familiar representations, such as algebra tiles, can be used to give access to the same underpinning structure.

Students need to	**Guidance, discussion points and prompts**
Identify growth in quadratic sequences *Example 1:* *The counter below is the starting point for a pattern.*  *Amrit adds some counters to make a 2-by-2 square of counters, shown below. There are 2^2 or 4 counters in total.*  *Ben adds counters to make a 3-by-3 square, shown below. There are 3^2 or 9 counters in total.*  *Callum adds counters to make a 4-by-4 square, shown below. There are 4^2 or 16 counters in total.*  *How many counters need to be added to:*	Moving from one term to the next in a quadratic sequence does not give a constant difference. *Example 1* uses counters arranged in an array to consider the term-to-term growth of the square numbers. Students may have only worked with linear sequences at Key Stage 3 and therefore only experienced constant rates of growth. A key aim is, through these examples, **deepening** students' understanding that the term-to-term growth of the square numbers is not constant, but there is a regularity to the growth. The shading in the **representation** draws attention to this growth pattern. Point out to students the way in which an extra row of counters is added to square, and an extra column of counters is added to the square, and then one more counter is added to 'fill the gap'. Students may 'see' this growth in different ways; explore these, ensuring that all the images show growth of $2n + 1$. Making the distinction between 'where the square is', 'where the linear pattern is', and 'where the constant is', will help students to notice the three separate elements that contribute to a quadratic sequence. Draw attention to the way that the number of black and checkered counters grow at each step: the grey make the $2n$ and the checkered can be thought of as the $+1$. Students should then be able to use their knowledge from Key Stage 3 to identify that the growth has a constant difference and so is linear. Identifying that the growth around a square grows by a constant amount offers an insight into why the second difference of a quadratic sequence is constant. Pay attention to the **language** used to describe the growth inherent within subsequent terms of this sequence, as subtly different meanings need to be conveyed. Students should be supported to realise that the black counters represent a linear sequence and therefore have a constant rate of growth. The white counters have a changing rate of growth, but the rate of change is constant. The growth of a

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a) *A 10-by-10 square to turn it into an 11-by-11 square?*

b) *A 23-by-23 square to turn it into a 24-by-24 square?*

c) *An n-by-n square to turn it into an $n+1$ by $n+1$ square?*

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growth pattern is often a challenging concept for students to understand, but is fundamental to understanding how difference patterns work.

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Example 2:

Dan and Effie are trying to find the rule for the nth term of the square numbers.

<table>
<tr><td>square numbers</td><td>1</td><td>4</td><td>9</td><td>16</td><td>25</td><td>36</td><td>...</td></tr>
<tr><td>difference</td><td>+3</td><td>+5</td><td>+7</td><td>+9</td><td>+11</td><td>+...</td><td></td></tr>
</table>

Dan says, 'To find the nth term of a sequence, I need to find the constant difference. The difference for square numbers isn't constant, so there can't be a rule.'

a) *Do you agree with Dan?*

*Effie says, 'There is a rule. The difference increases by 2 each time, so the **difference of the difference** is always 2.'*

b) *Do you agree with Effie?*

c) *Predict the next two differences. Do they generate the next two squares?*

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Example 2 uses the familiar **representation** of a table of differences to draw attention to the constant second difference. Students should be supported to move from the pictorial to numerical representations. Reference back to where each of the values on the table can be 'seen' in the previous images of the counters may help with this. For example, the row of differences can be found in the black and checked counters that are being added each time.

Working with different representations at the same time like this, supports students in making connections and **deepening** understanding of the ideas being developed. To support this, *Examples 1* and *2* both use the same values, although *Example 2* extends the sequence further. Make the connections between the two examples more explicit by asking students to sketch or describe what the growth from 25 to 36 counters might looks like. How would they be shaded if *Example 1* included the 6^2?

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Example 3:

Gwen writes out two consecutive square numbers.

The difference between them is 9.

a) *What are Gwen's square numbers?*

What would the square numbers be if the difference was:

b) *13?*

c) *23?*

d) *33?*

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Example 3 makes explicit that the difference between consecutive terms of a quadratic sequence grows by $2n+1$, thus **deepening** students' understanding of the process to find the nth term of a quadratic sequence. Part a, where the difference is 9, can be solved by looking back through *Examples 1* and *2*. Part b requires students to use their understanding of the nth term to reach an answer.

Given that finding the nth term of quadratic sequences is often taught procedurally, without drawing attention to the underlying structure, colleagues may have different opinions about the value of using examples such as these with students. While the reality is more complex, offering a binary choice to discuss may raise useful issues and highlight the complexity of decisions being made. For example:

'Which of the statements A and B do you align with most?

A: Example 3 is a useful way to deepen students understanding of a complicated procedure.

B: Example 3 is an unnecessary complication to an already complicated procedure.'

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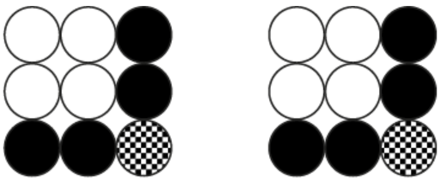
Appreciate that, in a quadratic sequence, the second difference is constant, and is double the coefficient of n^2

Example 4:

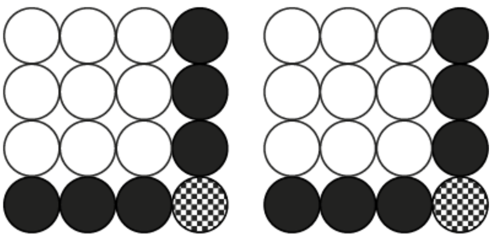
These counters are the starting point for a pattern.

Alma makes two 2-by-2 squares of counters, shown below. There are 2×2^2 *or 8 counters in total.*

Beth adds counters to make two 3-by-3 squares, shown below. There are 2×3^2, *or 18 counters in total.*



Cassie adds counters to make two 4-by-4 squares, shown below. There are 2×4^2, *or 32 counters in total.*



How many counters need to be added to:

a) *Two 10-by-10 squares to turn them into two 11-by-11 squares?*

b) *Two 23-by-23 squares to turn them into two 24-by-24 squares?*

c) *Two* n*-by-*n *squares to turn them into two* $n+1$ *by* $n+1$ *squares?*

Examples 4 and *5* mirror the structure and **representations** used in *Examples 1* and *2*, building on the familiarity to further secure students' understanding of the mathematical structure that underpins quadratic sequences.

In this example, **variation** is used to make explicit the change in the results when working with the sequence $2n^2$ rather than n^2.

Working with *Examples 4* and *5* on the board together may support students in making connections and **deepening** their understanding. Consider how to connect *Examples 1* and *4*, and *Examples 2* and *5*, since these offer the same task, with a different coefficient of n^2. Encourage students to generalise from here, and to consider the way in which the second difference is related to the coefficient of the squared term in a quadratic sequence.

Reflect on your teaching of quadratic sequences. Do students understand the relationship between the second difference and the quadratic nature of the sequences? Is it something that can be built upon or is it an accepted fact?

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Example 5:

Della and Eddie are trying to find a rule for the sequence made by doubling the square numbers.

<table>
<tr><td>2 x square numbers</td><td>2</td><td>8</td><td>18</td><td>32</td><td>50</td><td>72 …</td></tr>
<tr><td>difference</td><td></td><td>+6</td><td>+10</td><td>+14</td><td>+18</td><td>+22</td></tr>
</table>

Della says, 'There's no rule that I can see here.'

*Eddie points out that the difference is increasing by four each time and says, 'The **difference of the difference** is always four.'*

Do you agree with Eddie or Della?

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The **language** used by Eddie in *Example 5* mirrors that of Effie in *Example 2*. Consider at what stage you will develop Eddie and Effie's concept of 'the difference of the difference' to the language of 'second difference' more commonly used when working with quadratics.

At this stage, where students have a developing sense of the structures involved, asking them to generate their own sequences may be helpful for **deepening** their understanding. Ask them to give examples of what the first differences may be if the second differences are, for example, six or three instead. What would the quadratic sequence look like in these cases?

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Example 6:

Fred writes out a sequence that is made by multiplying the square numbers by a number, a.

First, he decides to use $a = 3$ and writes 3, 12, 27, 48, 75.

a) What are the first three terms she writes when $a = 6$?

In one of the sequences Fred writes, the difference of the differences is 10.

b) What is the value of a in this sequence?

In another sequence, Fred uses the rule $8n^2$ to generate the sequence.

c) What will the difference of the differences be?

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Example 6 offers practice through which students can rehearse the ideas developed to this point. This practice should support students in **deepening** their understanding by attending to the same relationship but in different ways in parts a, b and c.

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Example 7:

Parts of two sequences of the form mn^2 include the consecutive terms given below. What is the value of m in each of these sequences?

a) … 75, 108, 147, 192, …

b) … 80, 125, 180, 245, …

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As with *Example 6*, *Example 7* gives an opportunity for **deepening** students' awareness of the connection between the coefficient of n^2 in the rule for the nth term, and the second difference in the terms of a quadratic sequence.

Consider classroom practices that allow this relationship to be the focus of student tasks, rather than focusing on collecting correct answers.

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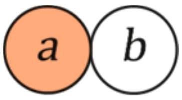
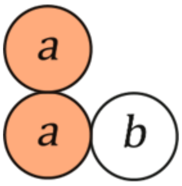
Example 8:

Logan is modelling linear sequences with some labelled counters.

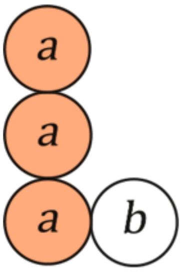
Term 1



Term 2



Term 3



a) *What do you notice about the different ways that a and b grow each time?*

Logan says, 'My counters could represent any linear sequence $an + b$. The a tile is the coefficient of n and the b tile is the constant.'

Izzy finds an example of a linear sequence.

153, 190, 227, 244, …

Logan then writes:

$2a + b = 190$

b) *Explain how he has generated his equation.*

c) *Write some more equations using the expressions for terms in this sequence.*

d) *Choose two of your equations and solve simultaneously to find values for a and b.*

e) *Find the nth term for Logan's sequence using a familiar method. What do you notice about this compared with your answer to part c?*

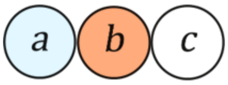
Example 8 is a step away from quadratic sequences and is not designed as a stand-alone task. Rather, it establishes the **representations** and conceptual framework to enable students to understand how to find the nth term for a quadratic sequence using an algebraic approach. The intention is to seed the idea that expressions can be written to describe terms within a sequence, and that these expressions can form equations using the value of each term. Once more than one equation has been written, students can use their knowledge of solving simultaneous equations to generate the nth term.

Consider *Example 8* and *Example 9* in the context of the students you are working with. Is there always a benefit to exploring a representation or method with a linear sequence before a quadratic sequence? Could *Example 9* be used without *Example 8*?

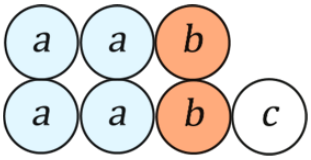
Example 9:

Logan uses his labelled counters to grow a sequence in a different way.

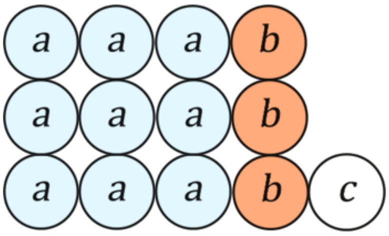
Term 1



Term 2



Term 3



a) *What do you notice about the different ways that a, b and c grow each time?*

Logan says, 'I know each term in the sequence is $an^2 + bn + c$.'

He then writes:

Term 1	$a + b + c$
Term 2	$4a + 2b + c$
Term 3	$9a + 3b + c$

b) *Where do the coefficients of a, b and c come from?*

Izzy finds an example of a quadratic sequence:

6, 17, 34, 57, …

Logan says, 'I can use simultaneous equations to find the answer.'

He writes:

Term 1	$a + b + c = 6$
Term 2	$4a + 2b + c = 17$
Term 3	$9a + 3b + c = 34$

Example 9 builds on *Example 8* to demonstrate an algebraic approach to finding the nth term of a quadratic sequence by linking the second difference and coefficient of n^2. It offers a justification for why that is the case rather than suggesting a series of steps without conceptual understanding. The **representation** of the counters is intended to offer a visual support to explain how the expressions for each term are generated. Organising this information into a table such as the one below may help students to understand the structure of finding differences in this context:

Term	1	2	3	4
Expression	$a+b+c$	$4a+2b+c$	$9a+3b+c$	$16a+4b+c$
Difference 1	$3a+b$	$5a+b$	$7a+b$	
Difference 2	$2a$	$2a$		

In the classroom, a teacher might want to change the way 'Logan and Izzy' approach this question to align with their departmental thinking around simultaneous equations. Students' attention should be drawn to the coefficients of a, b and c and how they change or remain constant. Systems of three simultaneous equations can be a challenge so offering a scaffolded approach allows students to focus on the intended key points. Teachers can choose which steps Logan and Izzy complete, and which steps are left to students, to ensure that the emphasis remains on **deepening** conceptual understanding.

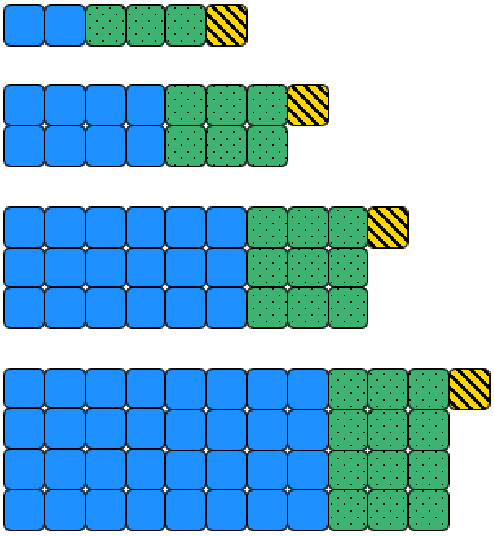
You might want students to explore this approach to fining the nth term with other quadratic sequences; consider the **variation** in the choice of sequences carefully, so that differences in example do not distract from the similarities in structure that you are trying to elicit.

Examples 8 and *9* both use a series of prompts to narrate two fictional students' lines of thinking. This breaks down an intimidating process into a series of logical steps. Reflect on the steps provided here: is it the appropriate level of scaffolding for your students? Work with your department to rewrite the examples with fewer supporting steps, and with more. Which version do you feel is most effective in ensuring your students both readily understand the mathematical structure, and need to think sufficiently hard?

c) *Explain how Logan has formed these equations.* *Logan says, 'The difference between term 1 and term 2 is* $3a + b = 11$*.'* *Izzy says, 'The difference between term 2 and term 3 is* $5a + b = 17$*'.* d) *What might Izzy and Logan do next?* *Logan says, 'If I know that* $a = 3$*, I can get rid of the quadratic part and just work out the* n*th term normally.'* e) *What do you think?*	
Appreciate that all quadratic sequences can be considered as a combination of a quadratic and linear sequence *Example 10:* *Gwen writes the first five terms of a sequence using the rule* an^2*. Hina writes the first five terms of a sequence* $2n + 7$*.* *Jason adds each of the terms of Gwen's sequence to the corresponding term in Hina's sequences and produces one sequence.* *The sequence that Jason writes is:* *12, 23, 40, 63, 92, …* a) *Write the first five terms of Hina's sequence.* b) *Work out the first five terms of Gwen's sequence.* c) *What is the value of* a *in Gwen's sequence?* d) *Use this information to write a rule for the* n*th term of Jason's sequence.*	*Example 10* builds on the ideas developed through the previous examples. It is **deepening** students' awareness that, having identified a sequence as quadratic because of the constant second difference, any quadratic sequence can be deconstructed into a linear sequence and a simple, single term quadratic sequence. That is, a sequence of the form $an^2 + bn + c$ can be thought of as the sum of two distinct sequences, an^2 and $bn + c$. By using the second difference to calculate the value of *a*, the sequence $bn + c$ can then be calculated and the values of *b* and *c* found.

Example 11

Basit and Serpil are looking at this pattern:



a) *Describe the growth in the striped yellow-and-black tiles.*

b) *Describe the growth in the spotted green tiles.*

c) *Describe the growth in the plain blue tiles.*

Basit says, 'The spotted green tiles have three more each time.'

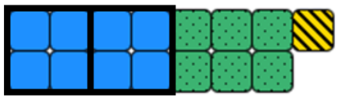
d) *Can you see what Basit means? Where can you see the extra three each time?*

Serpil says, 'The plain blue tiles are like two squares growing each time.'

e) *Can you see what Serpil means? Where can you see the two squares each time?*

f) *Describe the plain blue, spotted green and striped yellow-and-black sequences algebraically.*

Example 11 builds on *Examples 1* and *4* above. The **representation** emphasises the different parts of the sequence, inviting students to consider the quadratic, linear and constant elements separately. Time may be needed to connect the representation to the structure of the sequence. If students struggle to 'see' the growth of $2n^2$ then it may be helpful to highlight n^2 in each term as shown for the first two terms below:



By considering general quadratic sequences, students will be **deepening** their understanding of the sequences, particularly why the quadratic element dominates the rate of growth.

In this example. we are considering the distinction between 'where the quadratic pattern is', 'where the linear pattern is' and 'where the constant is' to help students to understand how the three separate elements contribute to a quadratic sequence. Students will need to be confident with using the necessary mathematical **language** to distinguish between each element, otherwise there is a risk of confusion. It is also important that students see the growth in each part then combine them. Support this by looking at each part and then the sequence as a whole. It might be useful to consider the sequence as:

$$2+3+1$$

$$8+6+1$$

$$18+9+1$$

$$32+12+1$$

so that students can see which elements contribute to the first and second differences.

It is likely to be helpful for links to be made with quadratic functions, and to prepare students for maths beyond Key Stage 4. Discuss how you make the structure of quadratic nth terms explicit to students. Is there a clear rationale for the procedure being used? Is it worth exploring the curriculum links to simultaneous equations and functions at this point in their learning? Quadratic sequences may not seem like a large part of an exam specification, but the links with other areas of maths are numerous.

Example 12:

Caitlin is trying to find the general term of the quadratic with

$$u_1 = 0, u_2 = 3, u_3 = 10, u_4 = 21$$

a) *How would you do this?*

Caitlin writes the following:

0, 3, 10, 21 …

+3 +7 +11

+4 +4

Second difference is 4, so use $2n^2$

Sequence: 0 , 3 , 10 , 21

$2n^2$: 2 , 8 , 18 , 32

Change: −2 , −5 , −8 , −11

−3 −3

−2 −(−3) = 1

Change is −3n + 1

∴ Sequence is

$2n^2 - 3n + 1$

b) *Describe what Caitlin has done each step of the way.*

c) *Compare her method to:*

$$u_n = 2n^2 - 3n + 1$$

Throughout these examples, there has been a range of complementary approaches to finding the nth term of a quadratic sequence. Other approaches may work well depending on setting and context. In *Example 12*, ‘Caitlin’ uses a typical method to find the nth term of a quadratic sequence. How would experience of the previous examples contribute to **deepening** students’ conceptual understanding as they explore Caitliln’s version of the procedure?

There are multiple valid approaches to this, and it is useful to see how students instinctively tackle it in part a. It might be helpful to reveal a line at a time and ask students to explain what has happened and what might happen next.

It is worth considering the sequence of learning that students experience when learning the nth term of a sequence. The new key ideas explored within this document should follow coherently from earlier experiences. Students should have the potential to build on these ideas beyond Key Stage 4. Take the opportunity to consider how your curriculum planning supports the transition from linear to non-linear sequences. Do students have a coherent experience of sequences through Key Stages 3 and 4? How do you build on learning from Key Stage 2? How do you ensure that concepts are established securely for further study at Key Stage 5?

Appreciate the difference between growth in a polynomial and geometric sequences

Example 13

Arlo is working out the differences between terms in sequences. He notices that, for n^3, the third row of differences is constant:

1 8 27 64 125

7 19 37 61

12 18 24

6 6

Example 13 could arguably belong in the previous set of exemplified key ideas, 9.2.1.3, as it is **deepening** students’ understanding of exponentiation. However, it is included here as the question is structured around exploring the differences between terms in polynomials, and so knowledge of at least some of the previous examples is essential. By comparing these differences with the ever-increasing differences between terms in the sequence 2^n, students should come to appreciate that the difference between terms in a geometric sequence is never constant.

The idea being used to provide this contrast is that, for a polynomial of degree n, the n^{th} rate of change will be constant. (Note that the **language** of ‘polynomial’ is not introduced until Key Stage 5; it is used here as a generalisation to support discussions with colleagues.) Students can use familiar language to generalise by noticing the pattern that a quadratic has a constant second difference, a cubic a constant third, a quartic a fourth, and so on. This is not true of exponential functions (or trigonometric functions) as the rate of change continually

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He wonders if this always happens for n^3 and whether the same thing happens with n^2 (1, 4, 9, 16, 25…) and n^4 (1, 16, 81, 256, 625…).

a) Work out the differences for the terms in these sequences until you arrive at a constant difference. What do you notice?

b) What happens for $n^3 + n^2$?

c) What happens for $3n^2$?

d) What would you expect to happen for:

(i) n^5
(ii) n^8
(iii) n^{25}?

Arlo now considers the sequence with the nth term 2^n: 2, 4, 8, 16, 32…

e) What is different about this sequence?

When will the difference be constant?

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varies. This concept can be hard for students to truly comprehend.

How deeply do you currently explore rates of change with Key Stage 4 students? Is there space in your curriculum plan to explore this, and perhaps to make links to topics like functions, kinematics and compound interest?

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Collaborative planning

Although they may provoke thought if read and worked on individually, the materials are best worked on with others as part of a **collaborative professional development** activity based around planning lessons and sequences of lessons.

If being used in this way, is important to stress that they are not intended as a lesson-by-lesson scheme of work. In particular, there is no suggestion that each key idea represents a lesson. Rather, the fine-grained distinctions offered in the key ideas are intended to help you think about the learning journey, irrespective of the number of lessons taught. Not all key ideas are of equal weight. The amount of classroom time required for them to be mastered will vary. Each step is a noteworthy contribution to the statement of knowledge, skills and understanding with which it is associated.

Some of the key ideas have been extensively exemplified in the guidance documents. These exemplifications are provided so that you can use them directly in your own teaching but also so that you can critique, modify and add to them as part of any collaborative planning that you do as a department. The exemplification is intended to be a starting point to catalyse further thought rather than a finished ‘product’.

A number of different scenarios are possible when using the materials. You could:

- Consider a collection of key ideas within a core concept and how the teaching of these translates into lessons. Discuss what range of examples you will want to include within each lesson to ensure that enough attention is paid to each step, but also that the connections between them and the overall concepts binding them are not lost.
- Choose a topic you are going to teach and discuss with colleagues the suggested examples and guidance. Then plan a lesson or sequence of lessons together.
- Look at a section of your scheme of work that you wish to develop and use the materials to help you to re-draft it.
- Try some of the examples together in a departmental meeting. Discuss the guidance and use the PD prompts where they are given to support your own professional development.
- Take a key idea that is not exemplified and plan your own examples and guidance using the template available at Resources for teachers using the mastery materials | NCETM.

Remember, the intention of these PD materials is to provoke thought and raise questions rather than to offer a set of instructions.

Solutions

Solutions for all the examples from *Theme 9 Sequences, functions and graphs* can be found here:

https://www.ncetm.org.uk/media/23eejt3r/ncetm_ks4_cc_9_solutions.pdf