7 Using and applying numerical structure

Mastery Professional Development

7.2 Using structure to transform and evaluate expressions

Guidance document | Key Stage 4

Using these materials

Click the heading to move to that page. Please note that these materials are principally for professional development purposes. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

Making connections

Building on the Key Stage 3 mastery professional development materials, the NCETM has identified a set of five 'mathematical themes' within Key Stage 4 mathematics that bring together a group of 'core concepts'.

The first of the Key Stage 4 themes (the seventh of the themes in the suite of Secondary Mastery Materials) is *Using and applying mathematical structure*, which covers the following interconnected core concepts:

7.1 Using structure to calculate and estimate

7.2 Using structure to transform and evaluate expressions

This guidance document breaks down core concept 7.2 Using structure to transform and evaluate *expressions* into three statements of **knowledge, skills and understanding**:

7.2 Using structure to transform expressions

- 7.2.1 Explore the structure and arithmetic of indices
- 7.2.2 Explore the structure and arithmetic of fractions and surds
- 7.2.3 Construct and critique algebraic arguments and proofs

Then, for each of these statements of knowledge, skills and understanding we offer a set of **key ideas** to help guide teacher planning:

- 7.2.1 Explore the structure and arithmetic of indices
  - 7.2.1.1 Understand the structures that underpin multiplication and division with numbers written in index notation
  - 7.2.1.2 Use the structures underpinning multiplication and division of numbers written in index notation to understand negative and zero indices
  - 7.2.1.3 Understand the structures that underpin raising a number written in index notation to a further index
  - 7.2.1.4 Use the structures involved when working with numbers written in index notation to understand fractional indices
  - 7.2.1.5 Use and apply the understanding of structures involved when working with numbers written in index notation to a range of problems
- 7.2.2 Explore the structure and arithmetic of fractions and surds
  - 7.2.2.1 Understand the dual meaning of a fraction as both an operation and the result of that operation
  - 7.2.2.2 Introduce the notion of a surd, as an operation and the result of that operation
  - 7.2.2.3 Revisit the structures underpinning the addition and subtraction of fractions (with a view to generalising the form)

7.2.2.4 Understand the structure that underpins addition of surds, for example $\sqrt{a}+\sqrt{a}=2\sqrt{a}$

7.2.2.5 Revisit the structures underpinning equivalent fractions (with a view to generalising the form)

7.2.2.6 Revisit the structures underpinning the multiplication and division of fractions (with a view to generalising the form)

7.2.2.7 Understand the structure that underpins multiplication of surds, for example $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$

7.2.2.8 Understand and use the technique of rationalising the denominator to transform a fraction to an equivalent fraction

7.2.3 Construct and critique algebraic arguments and proofs

7.2.3.1 Understand and use fluently general algebraic forms, e.g. an as expressions of multiples, $2n+1$ as even numbers etc.

7.2.3.2 Make and test conjectures about the generalisations that underlie patterns and relationships

7.2.3.3 Appreciate what constitutes the proof of a statement and what is required to disprove it

7.2.3.4 Understand how to describe a situation using algebraic symbols

7.2.3.5 Be able to manipulate and transform algebraic expressions, including cubics, in order to prove or show a result

7.2.3.6 Be able to use algebra and algebraic manipulation to construct logical arguments

Overview

This core concept explores indices, fractions, roots and surds more deeply, so that students can understand and manipulate expressions involving values in these forms. It considers the fact that numbers can be written in different forms, and that representing a number in an alternative form can make calculating more efficient. The intention is that students are confident not only in simplifying such expressions but also in converting them to alternative, equivalent forms, so that they can see connections and prove certain results.

As with the first two themes of the Key Stage 3 PD materials, '*1 The structure of the number system*' and '*2 Operating on number*', there is an emphasis on generalising the structure of number and number operations. So, rather than seeing algebraic manipulation as a separate skill to learn, it is embedded in students' understanding of numerical structure. This enables new learning, such as the laws of indices or rationalising the denominator, to be understood deeply rather than merely memorised as a series of steps.

Students will have met index notation through their work on square and cube numbers in Key Stages 2 and 3, but the work at Key Stage 4 involves a more in-depth appreciation of the structure of indices in general and the laws of arithmetic which they obey. This work signals the introduction of a new mathematical structure, that of exponentiation – or repeated multiplication. As in Key Stages 2 and 3 it was important for students to appreciate the difference between additive and multiplicative structures (i.e., between addition and repeated addition), so here at Key Stage 4 students need to appreciate the difference between multiplication and repeated multiplication.

Central to work with both indices and surds is the idea of a 'procept'. This is a mathematical expression which represents both a signal to operate, and the result of that operation.

A common error is to equate 'roots' with 'surds', instead of recognising that where an exact value for a root cannot be found, it is irrational and is known as a surd. Students' experience up until this point will have been calculating square roots, so they might equate $\sqrt{49}$ with 7 and see a surd as the process of 'square rooting'. So, a natural response to seeing $\sqrt{2}$ might be to perform the operation 'take the square root of 2', either using a calculator or estimation, to gain 'the answer'. However, a key awareness in this core concept is that $\sqrt{2}$ is also the name of the **exact** number which is 'the square root of 2' and, therefore, can be operated on in this form. As such, carrying out 'calculations' with surds is very akin to manipulating and simplifying algebraic expressions, where a surd can, much like x, be thought of a single term which can be operated on. It is also important to recognise that any calculation that 'turns' a surd into a decimal is by necessity an approximation: 1.414 is an approximate value for $\sqrt{2}$.

This idea of a procept is also useful when considering fractions, for example, $\frac{5}{16}$ can be thought of as a way of writing the calculation $5 \div 16$, and is also the answer to that calculation. In fact, this calculation may well have been used to find its decimal equivalent of 0.3125. It is not uncommon for students to prefer the decimal equivalent or even feel that the decimal is the value of the fraction. However, as with the case of $\sqrt{2}$, $\frac{5}{16}$ is the exact result of the calculation it is describing. This is crucially important for students to appreciate, particularly when dealing with fractions such as $\frac{1}{3}$ or $\frac{2}{7}$ where there is no terminating decimal which expresses the exact value.

Prior learning

Students will have first been introduced to the idea of indices towards the end of Key Stage 2, when they began to work with square and cube numbers. During Key Stage 3, they will have consolidated this learning, for example by working further with squares and cubes, including in contexts such as area, volume and Pythagoras' theorem. They will also have learnt that exponents take priority over the four operations in calculations, and started to manipulate algebraic expressions where terms are exponents. They are particularly likely to have worked with powers of 10.

Similarly, students' experience of roots will have begun when working with square numbers. They will have used the notation to express square roots, initially of square numbers, then of other integers, and this will then have developed through work on area and the use of Pythagoras' theorem to find the lengths of sides of 2D shapes. Initially, students tend to be uncomfortable leaving answers in root form rather than using a calculator to obtain a decimal 'answer', and so the understanding that the root form is exact whereas the decimal is an approximation is important.

Throughout Key Stage 3, students will have formalised the algebraic thinking begun in primary school and started working with the rules and conventions of algebraic notation. They should be able to manipulate equations so that they are in a form that is ready to solve, and understand where expressions are equivalent (for example, where a single term has been taken out as a factor). Many of the aspects of early proof at Key Stage 4 draw on knowledge that has been introduced much earlier in the curriculum. Rules around calculating with odd and even numbers, for example, are introduced in Key Stage 1 and are used and developed throughout Key Stages 2 and 3. Similarly, rationalising the denominator and manipulating algebraic fractions draws upon knowledge of calculating with fractions that, while likely to have been consolidated over Key Stage 3, has its roots in Years 4 and 5.

All core concept documents within '*1 The structure of the number system'* and '*2 Operating on number'* from the Key Stage 3 PD materials explore the prior knowledge required for this core concept in more depth.

Checking prior learning

The following activities from the NCETM secondary assessment materials, Checkpoints and/or Key Stage 3 PD materials offer a sample of useful ideas for assessment, which you can use in your classes to check understanding of prior learning.

Reference	Activity
Checkpoints '*Expressions and equations*', 'Additional activity H: Moving twos'	Look at these three cards: a^2 $2a$ $a+2$ a) How would you read each of the cards out loud? b) Find a value for a that makes the red (left-hand) card have the greatest value. c) Find a value for a that makes the blue (right-hand) card have the greatest value. d) Find a value for a that makes the green (middle) card have the lowest value e) Find a value that makes all the cards have an equal value.
Secondary Assessment Materials, page 8	Put these numbers in order from smallest to largest: 1^9, 2^8, 4^5, 5^4, 7^3, 9^1. Explain how you decided.
Secondary Assessment Materials, page 11	Alice, Bekah and Clare are explaining why $\frac{2}{3} \div \frac{1}{3} = 2$. • Alice says, 'Because you turn the second number upside down and multiply, so $\frac{2}{3} \div \frac{1}{3} = \frac{2}{3} \times \frac{3}{1} = \frac{6}{3} = 2$.' • Bekah says, 'Because if I share two-thirds of a cake between one-third of a person then to get a whole person, I need to multiply by three, so that means that the person gets six-thirds of the cake, and six-thirds is the same as two.' • Clare says, '$\frac{2}{3} \div \frac{1}{3}$ means, "How many one-thirds are there in two-thirds?" Because two-thirds is the same as $2 \times \frac{1}{3}$, the answer must be two.' Which explanation do you find most convincing? Why?

Key vocabulary

Key terms used in Key Stage 3 materials

- cube root
- exponent
- square root
- standard index form

The NCETM's mathematics glossary for teachers in Key Stages 1 to 3 can be found here.

Key terms introduced in the Key Stage 4 materials

Term	Explanation
index laws	Where index notation is used and numbers raised to powers are multiplied or divided, the rules for manipulating index numbers. Examples: $2^a \times 2^b = 2^{a+b}$ $2^a \div 2^b = 2^{a-b}$
index notation	The notation in which a product such as $a \times a \times a \times a$ is recorded as a^4. In this example, the number 4 is called the index (plural indices) and the number represented by a is called the base.
irrational number	A number that is not an integer and cannot be expressed as a fraction $\frac{p}{q}$ for any integers p and q with a non-zero denominator. Examples: $\sqrt{3}$ and π Real irrational numbers, when expressed as decimals, are infinite, non-recurring decimals.
rational number	A number that is an integer or that can be expressed as a fraction whose numerator and denominator are integers, and whose denominator is not zero. Examples: -1, $\frac{1}{3}$, $\frac{3}{5}$, 9, 235 Rational numbers, when expressed as decimals, are recurring decimals or finite (terminating) decimals. Numbers that are not rational are irrational. Irrational numbers include $\sqrt{5}$ and π which produce infinite, non-recurring decimals.
surd	1. An irrational number expressed as the root of a natural number. Example: $3\sqrt{2}$ 2. A numerical expression involving irrational roots. Example: $3 + 2\sqrt{7}$

Key ideas

In the following list of the key ideas for this core concept, selected key ideas are marked with a . These key ideas are expanded and exemplified in the next section – click the symbols to be taken direct to the relevant exemplifications. Within these exemplifications, we explain some of the common difficulties and misconceptions, provide examples of possible pupil tasks and teaching approaches and offer prompts to support professional development and collaborative planning.

7.2.1 Explore the structure and arithmetic of indices

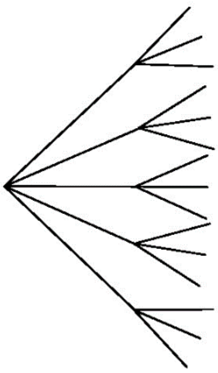
To calculate efficiently with integer indices, students should develop an understanding of numbers written in the form b^x, where the base number b is raised to an index x (also referred to as an exponent or power). Students should appreciate that the index laws can only be applied when there is a common base; drawing attention to numbers being of the same ‘unit’ and making connections with fractions having the same denominator will help students to understand this.

Students will be familiar with the different structures underpinning multiplication from Key Stages 2 and 3: scaling and repeated addition. A key aspect of this new work at Key Stage 4 is to appreciate the structure of exponentiation (repeated multiplication) where $7 \times 7 \times 7 \times 7 \times 7$ can be written more succinctly as 7^5. In fact, this index form can be thought of as a perfectly acceptable way of writing the answer to the multiplication: 7^5 is both the operation and the answer to the operation. Understanding that a^n is a number in its own right is key to being able to calculate with numbers in their index form.

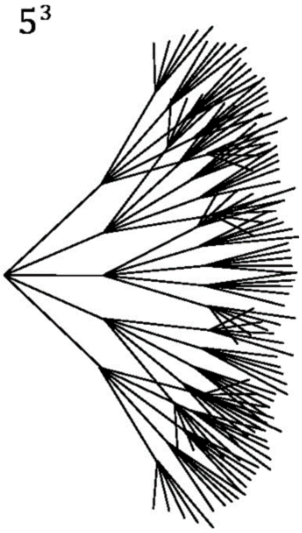
A common error, often due to students not having fully understood this new structure, is to confuse 7^5 and 7×5. Appreciating the difference in magnitude between $a \times b$ and a^b is a key understanding underpinning work on indices. It is important to give frequent opportunities for students to compare these two expressions, and other similar ones, to appreciate how very much larger one is than the other. For example, prompts such as, ‘*Which is greater,* 2^{10} *or* 10^2*?’* provide opportunities for greater depth of understanding.

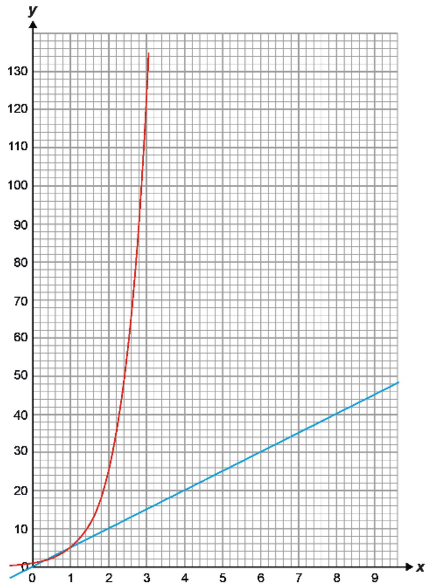
Representations can also help to highlight the difference between linear and exponential relationships, for example using images or graphs.

5×3



5^3





There are three principal laws of indices, which can be explored through writing the expanded form of the multiplications and divisions:

$$a^m \times a^n = a^{(m+n)}$$

$$a^m \div a^n = a^{(m-n)}$$

$$(a^m)^n = a^{mn}$$

Further experimentation with these structures can lead to an understanding of the derived rules, explored in exemplified key ideas 7.2.1.2 and 7.2.1.4:

$$a^0 = 1$$

$$a^{-m} = \frac{1}{a^m}$$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

7.2.1.1 Understand the structures that underpin multiplication and division with numbers written in index notation

7.2.1.2 Use the structures underpinning multiplication and division of numbers written in index notation to understand negative and zero indices

7.2.1.3 Understand the structures that underpin raising a number written in index notation to a further index

7.2.1.4 Use the structures involved when working with numbers written in index notation to understand fractional indices

7.2.1.5 Use and apply the understanding of structures involved when working with numbers written in index notation to a range of problems

7.2.2 Explore the structure and arithmetic of fractions and surds

To fully understand the meaning of surds, and how to calculate with them, students must have a deep understanding of some of the mathematical structures that they met in Key Stage 3, particularly in the context of fractions. Therefore, these materials revisit earlier work on fractions, ensuring the structures are fully generalised, and then connect it with the new Key Stage 4 work on surds.

For example, the idea of a fraction as a division $(a \div b)$ and as the answer to that division $\left(a \div b = \frac{a}{b}\right)$ is very similar to the notion of $\sqrt{2}$ being both an operation (take the square root of 2) and the answer to that calculation. This leads to the key understanding that any surd can be used in a calculation and manipulated like any other number or algebraic term. Students are then able to confidently simplify expressions such as $5\sqrt{3} - \frac{\sqrt{3}}{2}$ and leave their answer as a multiple of $\sqrt{3}$ without feeling the need to change it into a decimal. Indeed, they should understand that any decimal value generated by their calculator is an approximation of the more-accurate surd form. Similar reasoning should be applied to expressions with multiples of π.

The unitising structure behind the addition and subtraction of surds is exactly the same as that underpinning the addition of fractions. For example, $\sqrt{3} + 2\sqrt{3} = 3\sqrt{3}$ can be compared to a calculation such as $\frac{1}{5} + \frac{2}{5} = \frac{3}{5}$, as the ‘unit’ ($\sqrt{3}$ or fifths) must be the same for items to be added.

When rationalising the denominator of a fraction involving surds, students need to have generalised the idea that the value of any fraction will remain unchanged when multiplied by $\frac{p}{p}$. Students will have met this when finding equivalent fractions. In Key Stage 4, they will learn that to rationalise a fraction such as $\frac{1}{1-\sqrt{2}}$, using $p = 1 + \sqrt{2}$ will ensure that the denominator is rationalised. This method also relies on a deep understanding of the structure of the difference of two squares. Students need to recognise how the process of transforming $(a + b)(a - b)$ into $a^2 - b^2$ can be used to determine the value of p which will transform any given irrational denominator into a rational one.

7.2.2.1 Understand the dual meaning of a fraction as both an operation and the result of that operation

7.2.2.2 Introduce the notion of a surd, as an operation and the result of that operation

7.2.2.3 Revisit the structures underpinning the addition and subtraction of fractions (with a view to generalising the form)

7.2.2.4 Understand the structure that underpins addition of surds, for example $\sqrt{a} + \sqrt{a} = 2\sqrt{a}$

7.2.2.5 Revisit the structures underpinning equivalent fractions (with a view to generalising the form)

7.2.2.6 Revisit the structures underpinning the multiplication and division of fractions (with a view to generalising the form)

7.2.2.7 Understand the structure that underpins multiplication of surds, for example $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$

7.2.2.8 Understand and use the technique of rationalising the denominator to transform a fraction to an equivalent fraction

7.2.3 Construct and critique algebraic arguments and proofs

The idea of procept, introduced earlier in the context of fractions and surds, is particularly important when understanding and constructing algebraic arguments based on the manipulation of expressions of generality. An important awareness is that the expression $2m + 1$, for example, (where m is a positive integer) will always generate an odd number. However, students need to also appreciate that $2m + 1$ not only represents the operation 'double and add 1' but also the **result** when m is doubled and 1 is added. Then they will be able to see the expression $2m + 1$ as representing a general odd number and be able to use it as an expression which can be manipulated.

This powerful awareness allows for the translation of, for example, the phrase 'the sum of any two odd numbers' into the algebraic expression $(2m + 1) + (2n + 1)$. Factorising this expression means that it can then be transformed into $2m + 2n + 2 = 2(m + n + 1)$, and so prove that the sum of any two odd numbers is even.

It will be important for students to appreciate that such algebraic manipulation with the generalised form has allowed us to do arithmetic with **all** the odd numbers simultaneously. This therefore constitutes 'proof' of the statement, and to understand that this is more mathematically rigorous than merely showing it to be true in a few (or even a lot of) particular cases. This is a key understanding in students' maturing understanding of algebraic argument at Key Stage 4.

There are algebraic representations of numbers which are commonly used at this level, and there are situations when the decision made about which format is going to be used will affect efficiency. The skill to discern which format is best to use can only be developed through exposure to a wide range of examples.

7.2.3.1 Understand and use fluently general algebraic forms, for example an as expressions of multiples, $2n + 1$ as even numbers etc.

7.2.3.2 Make and test conjectures about the generalisations that underlie patterns and relationships

7.2.3.3 Appreciate what constitutes the proof of a statement and what is required to disprove it

7.2.3.4 Understand how to describe a situation using algebraic symbols

7.2.3.5 Be able to manipulate and transform algebraic expressions, including cubics, in order to prove or show a result

7.2.3.6 Be able to use algebra and algebraic manipulation to construct logical arguments

Exemplified key ideas

In this section, we exemplify the common difficulties and misconceptions that students might have and include elements of what teaching for mastery may look like. We provide examples of possible student tasks and teaching approaches (in italics in the left column), together with ideas and prompts to support professional development and collaborative planning (in the right column).

The thinking behind each example is made explicit, with particular attention drawn to:

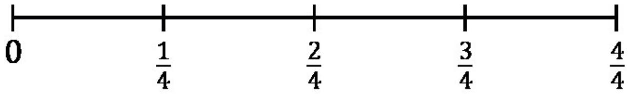
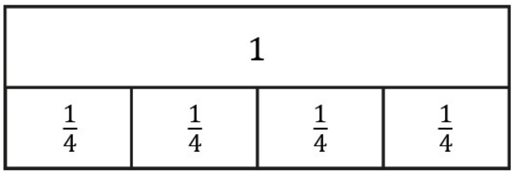
Deepening	How this example might be used for **deepening** all students' understanding of the structure of the mathematics.
Language	Suggestions for how considered use of **language** can help students to understand the structure of the mathematics.
Representations	Suggestions for key **representation(s)** that support students in developing conceptual understanding as well as procedural fluency.
Variation	How **variation** in an example draws students' attention to the key ideas, helping them to appreciate the important mathematical structures and relationships.

In addition, questions and prompts that may be used to support a professional development session are included for some examples within each exemplified key idea.

These are indicated by this symbol.

7.2.1.2 Use the structures underpinning multiplication and division of numbers written in index notation to understand negative and zero indices

Common difficulties and misconceptions
Students might readily accept and memorise formulae such as $a^0 = 1$ and $a^{-m} = \frac{1}{a^m}$ without understanding that they are necessary consequences of the principal law $a^m \div a^n = a^{(m-n)}$. Students need experience of using this principal law to reason why these formulae must be true. For example, time spent revisiting the simplification of fractions through finding common factors $\frac{12}{18} = \frac{3\times 4}{3\times 6}$ might clarify their understanding of 'cancelling' as finding common factors in the numerator and denominator and simplifying to 1. Students may also benefit from revisiting numbers written in standard form, for example $2 \times 10^{-3} = 0.002$, to expose the idea that $10^{-3} = 0.001$ or $\frac{1}{1000}$ or $\frac{1}{10^3}$. As with any index law, students with a less secure understanding might not appreciate that the law can only be applied when expressions share a base. Students should be exposed to examples and non-examples to ensure that they understand this. Useful links can also be made to their work on collecting like terms in Key Stage 3.

Students need to	Guidance, discussion points and prompts
Appreciate that divisions can be written as fractions and vice versa *Example 1:* a) *Provide a representation of the result of this calculation. And another…* $1 \div 4 = \_$ b) *Provide a representation of the result of this calculation. And another…* $1 \div 17 = \_$ c) *Provide a representation of the result of this calculation. And another…* $1 \div 100 = \_$	Students' familiarity with multiple **representations** of fractions may support them to make connections between fractions represented as numbers in different forms. It may be necessary to prompt students to draw different **types** of representations, as some may draw a circle divided into 4 and then a square divided into 4, etc.   Students should be prompted to explain their representation using specific **language** to encourage them to make links between the fraction as an operation and as a result. Ask questions such as, '*Where is the dividend in the representation? Where is the divisor in the representation?*' to prompt this thinking. Students will have encountered fractions from early in Key Stage 2. It is possible that their notion of fractions as instructions to operate may be more developed than their notion of fractions as numbers themselves, or vice versa. How might you assess whether they have a concept of **both** notions? What representations might support them if they need to work on one or other idea?
Example 2: *'The dividend in a calculation is the numerator in the result of the division.'* *Is this statement always, sometimes, or never true?*	Encourage students to use **language** that describes algebra as generalising and the general form, instead of as 'replacing numbers with letters'. For example, you could model this generalisation using statements such as, '*When the dividend a is divided by the divisor b, the quotient is the fraction $\frac{a}{b}$.*' **Deepen** students' thinking further by considering different cases for this statement. Prompt them to consider the cases where the dividend is an integer, where the result has been simplified, or where the dividend was a non-integer. Also, whether the type of number the divisor is has an impact. Consider your students' experience of writing fractions and divisions across the curriculum. When is it useful to represent fractions as divisions, and vice versa? Do students have opportunities to gain fluency in both formats?

Understand that simplifying fractions involves taking out a factor of 1 using the properties $\frac{ab}{ac} = \frac{a\times b}{a\times c}$ and $\frac{a}{a} = 1$ *Example 3:* a) *Simplify, by factorising the numerator and denominator, the fraction:* $$\frac{24}{48}$$ b) *Find another way of simplifying this fraction by factorising. And another…* c) *What do you notice about the results?*	Inspecting a fraction where the numerator and denominator share several factors is an opportunity for **deepening** students' understanding of the underlying structure. Given that 24 and 48 each have many factors, most of which are common, there will be many ways to simplify this fraction. Encourage students to notice that is it only when the highest common factor is extracted and simplified to 1 that the fraction is fully simplified. Algebraic **representations** may be used to draw students' attention to the general form. For example: the idea that $\frac{ab}{ac}$ is equal to $\frac{a\times b}{a\times c}$ and that $\frac{a\times b}{a\times c}$ is equal to $\frac{a}{a} \times \frac{b}{c}$ and is therefore also equal to $1 \times \frac{b}{c}$. Consider the choice of example and modelling. Is there a 'better' fraction to use than $\frac{24}{48}$? Would a fraction with far greater numbers for the numerator and denominator lead to a deeper understanding? How might the way in which the simplification is represented affect how students see this process?
Example 4: *Which is greater, $\frac{3}{3}$ **of** 7 or $\frac{7}{7}$ **of** 7?*	This example checks that students have an appreciation of the property that any number divided by itself is necessarily 1. If students are confident that a^m is a number and can be treated as such, they can make connections and recognise that $a^m \div a^m = \frac{a^m}{a^m}$. This leads to the derived principle that $a^0 = 1$. Some students may be ready to immediately consider the relevance of *Example 4* to the principle that $a^0 = 1$. Others might need more convincing, perhaps using *Example 5* or *Example 7*. How will you decide which is the case for your students?
Connect the understanding of simplifying fractions with simplifying expressions with indices *Example 5:* *For each of the following equations, suggest values for p, q and r. Is there more than one way to do this?* a) $\frac{48}{12} = \frac{p}{q} \times \frac{r}{r}$ b) $\frac{8\times8\times8\times8\times8\times8}{8\times8\times8\times8\times8\times8\times8\times8} = \frac{p}{q} \times \frac{r}{r}$ c) $\frac{5^3}{5^7} = \frac{p}{q} \times \frac{r}{r}$	*Example 5* makes explicit the intended learning point from *Example 3.* Here, the **representation** of the generalised form supports students to see the underlying structure. The **variation** in this example supports students to make connections: the same structure is used throughout, but the form of the numbers varies. Part *a* uses a fraction, so that students are coming from a place of familiarity from Key Stage 3. Part *b* uses the expanded form of the exponent, so that students are supported by the visual of each 8 in the multiplication and can see how many of these match up. Finally, part *c* uses index form, so that students are working with the new notation that they are learning at Key Stage 4.

Appreciate that a^n is a number in its own right, which can be operated on *Example 6:* *Write as fractions and then calculate, giving your answer in the form 13^a:* $13^{11} \div 13^6$ $13^{11} \div 13^7$ $13^{11} \div 13^8$ $13^{11} \div 13^9$ $13^{11} \div 13^{10}$ $13^{11} \div 13^{11}$ $13^{11} \div 13^{12}$ $13^{11} \div 13^{13}$ *What do you notice?*	Calculations such as $13^{11} \div 13^7$ can be considered as a transformation or **representation** of: $\frac{13 \times 13 \times 13 \times 13 \times 13 \times 13 \times 13 \times 13 \times 13 \times 13 \times 13}{13 \times 13 \times 13 \times 13 \times 13 \times 13 \times 13} = 13^4$ This process can be rewritten as: $13^{11} \div 13^7 = 13^{(11-7)}$ and later represented in the general form as: $a^m \div a^n = a^{(m-n)}$ Encourage students to use the **language** of simplifying the fraction, using the property $\frac{a}{a} = 1$, rather than 'cancelling'. If students have had an opportunity to consider how highest common factors can be used to simplify fractions, they can draw on this understanding to make connections as to why the second index can be subtracted from the first, **deepening** their understanding of this index law. Consider how you have introduced the index rule $a^m \div a^n = a^{(m-n)}$ to classes in the past. Do you think that it is better to share the rule and then offer a demonstration, or to demonstrate and then generalise? Discuss with your colleagues and share your experiences to help inform your schemes of work.
Use knowledge of simplifying fractions to appreciate why $a^0 = 1$ *Example 7:* *Mike says, '$2^4 \div 2^4 = 0$ because the two values cancel each other out.'* *Why is he wrong?*	Presenting students with a common misconception such as this offers an opportunity for **deepening** their understanding. Students need to be able to articulate why the two values here do not 'cancel' each other in the way that Mike suggests. If students find this challenging, prompt them to represent the division as a fraction and make connections with their work on equivalent fractions.
Example 8: a) *Complete each row in the table below with equivalent expressions. What do you notice?* b) *Is there more than one way to complete any column?* c) *Can you create a general rule for when a number is raised to a zero power?*	The **variation** in this example will help students notice how the value of 1 can be created by pairing up powers in the numerator and denominator, and therefore used to simplify the expression. Draw attention to pairs of rows and ask students, '*What is the same and what is different?*'

$3^5 \div 3^2$		$3 \times 3 \times 3 \times \frac{3 \times 3}{3 \times 3}$	3^3
	$\frac{3 \times 3 \times 3 \times 3}{3}$		3^3
$3^7 \div 3^5$			
		$3 \times 3 \times \frac{3 \times 3}{3 \times 3}$	
	$\frac{3 \times 3 \times 3 \times 3}{3 \times 3 \times 3 \times 3}$		3^0
$3^5 \div 3^5$			

Use knowledge of simplifying fractions to appreciate why $a^{-b} = \frac{1}{a^b}$

Example 9:

a) *Complete each row in the table below with equivalent expressions. What do you notice?*

b) *Is there more than one way to complete any column?*

c) *Can you create a general rule for when a number is raised to a negative power?*

As with *Example 8*, the **variation** in this example should help students notice how the value of 1 can be created by pairing up powers in the numerator and denominator, and therefore used to simplify the expression. As before, draw attention to rows which have equivalent values and ask students, '*What is the same and what is different?*'

$3^2 \div 3^5$		$\frac{1}{3 \times 3 \times 3} \times \frac{3 \times 3}{3 \times 3}$	3^{-3}
$3^5 \div 3^8$			3^{-3}
	$\frac{3}{3 \times 3 \times 3 \times 3}$		
		$\frac{1}{3 \times 3} \times \frac{3 \times 3}{3 \times 3}$	
$3^6 \div 3^8$			
			3^{-4}

Example 10: *Show that* $3^8 \div 3^{10} = \frac{1}{3^2}$	**Deepening** students' understanding of the index laws includes giving them opportunities to make connections between the primary principle and necessary consequence, rather than receiving the latter as a rule. For example, $\frac{a^3}{a^5}$ can be written as $\frac{a^3}{a^3} \times \frac{1}{a^2}$. This, alongside an appreciation that $a^3 \div a^5 = a^{(3-5)}$, can lead students to derive the principle that $a^{-m} = \frac{1}{a^m}$. Compare *Examples 9* and *10*. Each deconstructs the index law $a^{-m} = \frac{1}{a^m}$ in a slightly different way. When might you use each with your students? Is there a place for both in your curriculum?
Example 11: a^n *Freddie says, 'My expression has a value that is greater than 1.'* *Gurpreet says, 'My expression has a value that is less than 1.'* *Maya says, 'My expression has a value that is exactly 1.'* *If* a *is a positive integer, what do you know about the values that each person has chosen for* n*?*	*Example 11* offers an opportunity for **deepening** students' understanding by thinking about the properties of numbers that fulfil a particular statement. If students have a deep understanding of powers, then they should be able to identify that Gurpreet's value for n must be negative and that Maya's must be 0. Challenge further by asking students whether their responses would change if a did not have to be a positive integer.
Example 12: *Ms Hawkins draws a place-value table and asks three of her students to write the column headings from 100 to one thousandth using numerals. They choose three different ways of writing them.* *a) Complete each row of place-value column headings in the table below, using the form that the student has chosen.* *b) Could you write any of the headings a different way?*	Revisit the familiar **representation** of a place-value table to support students in making connections to negative indices. Noticing the relationship between, for example, the tenths and $\frac{1}{10^1}$, may support students to recognise indices as simply alternative ways of writing numbers. For example: $\frac{10^3}{10^5} = \frac{10^3}{10^3} \times \frac{1}{10^2} = 1 \times \frac{1}{10^2} = \frac{1}{100} = 0.01$ Compare this and the pattern of place-value column headings as: $10^3, 10^2, 10^1, 10^0, 10^{-1}, 10^{-2}$. It is helpful to draw students' attention to the alternative forms of the number 0.01 as both 10^{-2} and $\frac{1}{10^2}$. This is also an opportunity here to explore the **language** of place value and indices. Students need to have an appreciation that 0.01, 10^{-2} and $\frac{1}{10^2}$ are different ways of naming the same number: one hundredth.

Jane	100				$\frac{1}{100}$	
Pete	10^2					$\frac{1}{10^3}$
Sachin	10^2			10^{-1}		

Reason and problem solve using the laws of indices

Example 13:

These numbers decrease in size from left to right:

$$x^{-2}, x^0, x^1, x^4.$$

What is one possible value of x? Can you give another possible value of x which would also satisfy the criteria?

Example 13 is another opportunity for **deepening** students' understanding of index laws. Students will expect the values to increase in size from left to right, as the exponents increase. Presenting a situation that is the opposite of what they expect encourages them to consider the laws they have learnt. This should lead them to think about x being a non-integer value. Prompt students to define this precisely as a non-integer between 0 and 1. They should consider these questions:

- What are the properties of correct values of x?
- What is the range of correct values of x?

If you were to design a series of examples which had numerical values, which numbers would you choose? Why? Would a different set of numbers draw the students' attention to a different mathematical structure?

Example 14:

Find the value of x in each equation below:

a) $2^6 = 4^x$

b) $\frac{1}{16} = 2^x$

c) $\frac{1}{16} = 4^x$

d) $\frac{1}{81} = 3^x$

e) $9^{-2} = \frac{1}{3^x}$

f) $\frac{27}{243} = 3^x$

The **variation** in *Example 14* allows you to make comparisons and draw attention to how the same numbers can be expressed with different bases. Parts *b* and *c*, for example, use the same value but different bases. Students might use the relationship they established in part *a* to support them with this. Similarly, parts *d* and *e* use the same base, and the same value, but expressed in different ways, which should support students to make connections about the equality of these statements.

7.2.1.4 Use the structures involved when working with numbers written in index notation to understand fractional indices

Common difficulties and misconceptions
Students may have difficulty in understanding fractional notation when used as an exponent, so it is important to build the understanding step by step. Given that they should already understand the concept of an exponential number, they need to extend this knowledge, firstly to understand an exponent expressed as a unit fraction and then to move on to combine unit fraction and integer indices to obtain all other fractional indices. They are likely to encounter difficulties if their understanding of operations with fractions is insecure, so ensure that these foundations are in place. The second point of uncertainty arises when students don't know which part of a fractional exponent to tackle first. It is important that they realise that the numerator and denominator of a fractional exponent have equal precedence: it is a commutative operation. Working with some numerical examples to illustrate will help students to appreciate the commutativity.

Students need to	Guidance, discussion points and prompts
Appreciate that some numbers can be written as repeated multiplication in a number of different ways *Example 1:* a) *Rewrite 9^3 in the form 3^n, where n is an integer.* b) *Rewrite 9^4 in the form 3^n, where n is an integer.* c) *Rewrite 9^5 in the form 3^n, where n is an integer.* d) *Rewrite 3^{20} in the form 9^n, where n is an integer.* e) *Rewrite 3^{19} in the form 9^n, where n is an integer.*	Draw students' attention to the fact that 9^3 and 3^6 are two different **representations** of the same number. Consider how you might draw this understanding out. For example, write out 9^3 as $9 \times 9 \times 9$ and then ask students if there is another way to write the 9 in this expression. This question offers an opportunity for **deepening** students' understanding by enabling them to recognise the connection between 9^n and 3^{2n} . This example is designed with 9 and 3 as the bases to transform. Why do you think these numbers were chosen? Would a different set of numbers draw students' attention to a different mathematical structure? For example, how would this example be different if you used 16^3 and the form 4^n? Or 2^n?
Example 2: *If n is an integer, is it possible to express 16^3 in each of the forms given below?* a) 4^n b) 2^n c) 8^n d) 32^n e) 64^n *For each one that is possible, show the transformed expression.*	The **variation** in these examples has been designed to relate to the powers of 2. By keeping the initial expression the same, students' focus is on the different ways it can be transformed. This is to draw students' attention to the relationship between the bases and the indices. For parts *a* and *b*, students should be encouraged to notice how the 16 and the 4 are related, and how the 16 and the 2 are related, and then also how the 4 and the 2 are related. It may help students to think about different **representations** for the original expression. Writing it out using the smallest possible base (as twelve twos multiplied together) might help them to recognise which bases are possible with an integer exponent. Part *e*, 64^n, is included to prevent the misconception that the base needs to be smaller than in the original expression. The structure 'is it possible' supports with **deepening** students' understanding of the index laws. Asking how the question would change if it was not specified that n is an

<table>
<tr><td></td><td>integer opens further opportunity for deepening understanding. In this case, part d becomes possible.</td></tr>
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Example 3:

Compare these four strategies for calculating 512×64.

Greg

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<tr><td>×</td><td>500</td><td>10</td><td>2</td></tr>
<tr><td>60</td><td>30 000</td><td>600</td><td>120</td></tr>
<tr><td>4</td><td>2000</td><td>40</td><td>8</td></tr>
</table>

30 000
2 000
600
120
40
+ 8
32768

Bella

512×64
$= 8^3 \times 4^3$
$= (2^3)^3 \times (2^2)^3$
$= 2^9 \times 2^6$
$= 2^{15}$
$= 32768$

James

512×64
$= 8^3 \times 8^2$
$= 8^5$

Sanjay

512×64
$= 2^9 \times 2^6$
$= 2^{15}$

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In *Example 3,* students attend to strategies involving transforming the base and calculating in index form, rather than multiplying out in full numerical form. The question also offers an opportunity to consider different forms of the answer.

Discuss the **language** of 'calculate' and what that means. Encourage students to consider the equivalence of the 'different' answers. Discuss when each form might be most appropriate and encourage them to consider the benefits of calculating in index form for very large (and very small) numbers.

Consider how else you could design an example to support students to attend to transforming the base.

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Understand that $a^{\frac{1}{n}}$ is equivalent to $\sqrt[n]{a}$

Example 4:

a) *Calculate:*

$\sqrt{9} \times \sqrt{9}$
$9^{\frac{1}{2}} \times 9^{\frac{1}{2}}$
$\sqrt{25} \times \sqrt{25}$
$25^{\frac{1}{2}} \times 25^{\frac{1}{2}}$
$\sqrt[3]{8} \times \sqrt[3]{8} \times \sqrt[3]{8}$
$8^{\frac{1}{3}} \times 8^{\frac{1}{3}} \times 8^{\frac{1}{3}}$

b) *What do you notice about your answers? How might this help you to rewrite* $a^{\frac{1}{n}}$ *in another form?*

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In *Example 4*, two different **representations** of the same value are used each time to help students to understand the equivalence of the pairs of calculations. Students should recognise, from their work on key idea 7.2.1.2, that the fractional powers sum to 1. They should also, separately, know that the roots in each example will 'cancel' each other. The juxtaposition of the two should enable students to connect the square and cube roots with the denominators of the fractional powers.

These examples only use unit fractions as powers so that the focus can be on **deepening** students' understanding of writing roots in index notation. This understanding should be secure before moving on to fractions with different numerators and considering the combination of both roots and powers.

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Example 5:

Find values for a, b *and* c*:*

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Example 5 reinforces the learning from *Example 4*, so that students can show they understand exponents that are unit fractions are roots. The **variation** is such that, in

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$$\mathbf{64^{\frac{1}{a}} = 4}$$

$$\mathbf{64^{\frac{1}{b}} = 8}$$

$$\mathbf{64^{\frac{1}{c}} = 2}$$

Create your own set of equations using different fractional powers for the same base.

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keeping the base constant, students need to focus on what changes as the denominator of the exponent changes.

Asking students to create their own set of equations provides an opportunity for **deepening** understanding. Those who are less secure might choose any value as the base and rely on a calculator to generate ‘answers’. Prompt students to consider integer solutions; can they think of a value for the base that allows them to create more than one equation?

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Example 6:

a) *Express each number in the form $(a^m)^n$, where m is an integer > 1.*

<table>
<tr><th>Set A</th><th>Set B</th></tr>
<tr><td>$4^{\frac{1}{2}}$</td><td>$8^{\frac{1}{3}}$</td></tr>
<tr><td>$9^{\frac{1}{2}}$</td><td>$16^{\frac{1}{4}}$</td></tr>
<tr><td>$16^{\frac{1}{2}}$</td><td>$32^{\frac{1}{5}}$</td></tr>
<tr><td>$25^{\frac{1}{2}}$</td><td>$64^{\frac{1}{6}}$</td></tr>
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b) *How else can you write your results? What do you notice?*

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Example 6 offers an alternative strategy for **deepening** students’ understanding of exponents that are unit fractions. Key to this task is that m has to be an integer greater than 1, which means students cannot, for example, simply rewrite $4^{\frac{1}{2}}$ as $(4^1)^{\frac{1}{2}}$. Instead, they need to attend to how else 4 can be expressed. In this case, rewriting it as 2^2 leads to the expression $(2^2)^{\frac{1}{2}}$, which can be rewritten as $2^{\frac{2}{2}}$, 2^1 or simply 2. This is another way of showing that $4^{\frac{1}{2}}$ is equivalent to $\sqrt{4}$, demonstrating how the primary principle $(a^m)^n = a^{mn}$ relates to the derived principle $\sqrt[n]{a^m}$.

Compare *Example 6* with *Examples 4* and *5*. Which do you think would be most convincing for your students? Which is most accessible? Why?

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Understand that $\sqrt[n]{a^m} = a^{\frac{m}{n}}$ is a consequence of the primary principle $(a^m)^n = a^{mn}$

Example 7:

Find values for $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$:

$$\mathbf{(2^3)^2 = 2^a}$$

$$\mathbf{(4^b)^2 = 4^8}$$

$$\mathbf{(8^2)^{\frac{1}{3}} = 8^c}$$

$$\mathbf{(16^3)^d = 16^{\frac{3}{4}}}$$

Use your answers for $\mathbf{c}$ and $\mathbf{d}$ to explain why $\mathbf{8^{\frac{2}{3}} = \sqrt[3]{8^2}}$

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Example 7 builds so that students use existing knowledge to generalise about how the numerator and denominator of a fractional exponent interact. The **variation** considers the primary principle $(a^m)^n = a^{mn}$ with integer powers, before thinking about how that principle applies to fractions.

Students should understand:

$$\sqrt[n]{a^m} = (a^m)^{\frac{1}{n}} = a^{\frac{m}{n}}$$

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Example 8:

Explain why:

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Example 8 asks students to explain examples of the primary principle $(a^m)^n = a^{mn}$. What **language** might you expect students to use? Pay careful attention to the

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a) $(64)^{\frac{1}{3}} = \sqrt[3]{64}$ b) $(64^2)^{\frac{1}{3}} = \sqrt[3]{64^2}$ c) $(64)^{\frac{2}{3}} = \sqrt[3]{64^2}$ d) $(64)^{\frac{2}{3}} = \sqrt[3]{8^4}$ e) $(64)^{\frac{2}{3}} = \sqrt[3]{2^{12}}$	language they actually use. Do students have an appreciation of the equivalence of these two forms? The **variation** here focuses on students' developing fluency with manipulating exponents. Part *a* is relatively straightforward, with students simply needing to appreciate that unit fraction exponents can be written as roots. In parts *b* and *c* the fraction and exponent are combined. Parts *d* and *e* rely on students' familiarity with both powers of 2 and index laws to move between different bases. With colleagues, read out each of these examples in turn. Do you all 'read' them the same way? Why or why not? What might be the impact of any differences on students' understanding as they move between class teachers over their time at school?
Understand that applying the laws of indices is commutative *Example 9:* *Mara and Jon are trying to understand the notation* $x^{\frac{2}{3}}$ *Mara says, "Firstly, you have to find the cube root of* x *and then you have to square it".* *Jon disagrees and says, "You have to square* x *first and then find the cube root".* *Try Mara and Jon's approaches for the following numbers. What do you notice?* *a) 8* *b) 64* *c) 27* *d) 125* *Can you express algebraically what Mara and Jon are each doing?*	In *Example 9*, students are encouraged to notice that the order of operations does not matter when raising to a power or finding a root, since they are both indices with equal precedence. They may notice that for some numbers, one order proves easier than another. **Representing** each of the methods algebraically will help to understand the equivalence. Mara: $(x^{\frac{1}{n}})^m = x^{\frac{m}{n}}$ Jon: $(x^m)^{\frac{1}{n}} = x^{\frac{m}{n}}$ Students will have been developing their understanding of commutativity since their earliest experiences of addition in Key Stage 1. Check that they understand this **language** and that they can also apply the principle in the context of multiplication and exponentiation. Knowing that both Jon and Mara's approaches are equally valid demonstrates an understanding of the commutative law.
Appreciate the case when $(\boldsymbol{a^m})^{\boldsymbol{n}} = \boldsymbol{a^1}$ *Example 10:* *Consider the cube root of 8. The cube root of 8, cubed is, necessarily, 8. The same is true for other roots of 8.* *Calculate:* $\left(\sqrt[2]{8}\right)^2 = \_$ $\left(\sqrt[4]{8}\right)^4 = \_$ $\left(\sqrt[5]{8}\right)^5 = \_$ $\left(8^{\frac{1}{6}}\right)^6 = \_$	The equations in *Example 10* are presented in both root and index form, to check that students are confident with both **representations**. The intention is for students to come to accept the equivalence of writing the same expression in both root and fractional exponent forms. The calculations are prefaced with statements intended to support understanding. The **language** structures here may feel more applicable to the first three equations; discuss how to 'read' the fourth and fifth equations. Students should also be able to generalise that when an n^{th} root of any number is taken, that number can be restored by being raised to a power of the same order. You may find it helpful to prepare a sentence structure such as, *'When the* n^{th} *root is raised to the* n^{th} *power, the result will be the base.'*

$\left(8^{\frac{1}{n}}\right)^{n} = \_$ *Can you rewrite each example another way?*	$\underbrace{a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \dots}_{n\ times} = a$
Example 11: *Find a pair of values for m and n which satisfy $(7^m)^n = 7$.* *Find another pair… And another…*	*Example 11* draws on a similar idea to *Example 8*. Compare the two examples with colleagues. What is the same and what is different? What are the benefits and limitations of a more structured approach? When might you use each one?
Solve problems involving fractional indices *Example 12:* *Find values for $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$:* $(2^3)^2 = 8^a$ $(4^b)^2 = 2^{16}$ $(8^2)^{\frac{1}{3}} = 4^c$ $(32^3)^d = 16^{\frac{3}{4}}$	*Example 12* uses similar sets to *Example 7* but here the focus is on **deepening** understanding of manipulating the form of a number. In *Example 7* each value had the same base, whereas here students demonstrate that they can move between different bases while maintaining equality. Consider whether you would remind students of the original four equations before starting this task. Would that add or remove challenge?

7.2.2.7 Understand the structure that underpins multiplication of surds, for example $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$

Common difficulties and misconceptions

It is easy for students to accept the identity $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$ as a 'formula' to remember rather than something which expresses a fundamental structure of the arithmetic of surds.

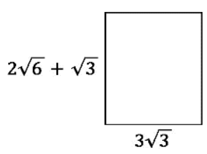
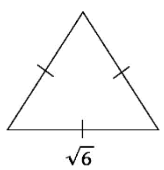
Spend time getting students to think about what happens when, for example, $\sqrt{2} \times \sqrt{3}$ is squared and compare this with when $\sqrt{6}$ is squared. Working with and discussing various examples like this can support students to generalise this relationship and realise that such a relationship exists whatever order the root is that is being taken (i.e., not just square roots).

An important realisation is that when one of the factors of a is a perfect square, $\sqrt{a}$ can be transformed into an expression of the form $b\sqrt{c}$. This enables students to transform, for example, $\sqrt{50}$ into $\sqrt{25 \times 2}$, which is equal to $\sqrt{25} \times \sqrt{2}$ or $5\sqrt{2}$. They are therefore able to simplify such expressions as $\sqrt{50} + \sqrt{2} + \sqrt{8}$ because each one can be written as a multiple of $\sqrt{2}$ and so the addition can be performed.

Students need to	Guidance, discussion points and prompts
Understand and use the equivalence of $\sqrt{a}\sqrt{b}$ and $\sqrt{ab}$ (as well as $\frac{\sqrt{a}}{\sqrt{b}}$ and $\sqrt{\frac{a}{b}}$) *Example 1:* *Calculate (or simplify) the following:* *a)* $(\sqrt{9})^2$ *b)* $(\sqrt{25})^2$ *c)* $(\sqrt{5})^2$ *d)* $(\sqrt{2})^2$ *e)* $\sqrt{5} \times \sqrt{5}$ *f)* $\sqrt{2} \times \sqrt{2}$ *g)* $(\sqrt{2})^2 \times (\sqrt{5})^2$ *h)* $\sqrt{2} \times \sqrt{2} \times \sqrt{5} \times \sqrt{5}$ *i)* $(\sqrt{2} \times \sqrt{5})^2$	The **variation** here stems from selecting examples with enough similarities to help students focus on the structures underpinning multiplication and squaring of square roots. The first four examples highlight the fact that squaring and taking the square root are inverses. Some students may write down intermediate calculations for parts *a* and *b* (for example, 3^2 and 5^2). Pause after these two questions and discuss whether this is necessary. This should draw attention to the fact that the operations are inverses of each other. Before moving on to part *e*, there is a further opportunity for **deepening** understanding in asking students to write some of their own examples of squaring a square root (or square rooting a square). Challenge them to be as creative as they can by using numbers in different forms or algebraic expressions, such as: $(\sqrt{1\,000\,000})^2$, $(\sqrt{\frac{5}{7}})^2$, $(\sqrt{p})^2$, $(\sqrt{3x+2})^2$, $(\sqrt{a^2+pq-7})^2$ In the subsequent examples, draw students' attention to the pairs of examples that are different **representations** of the same expression. For example, *c* and *e*, or *d* and *f*, demonstrate that $\sqrt{a} \times \sqrt{a}$ is another way of writing $(\sqrt{a})^2$. Through working on the remaining examples, the conclusion can be drawn that, in general, $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$. Consider where you think would be a good place to pause and have a whole-class discussion. What discussion prompts would you use?
Example 2: *Which of the below expressions is* ***not*** *equivalent to* $\sqrt{18}$*?*	It is important that students are aware of when expressions are different **representations** of each other, and when they are not. While most of the expressions in this set use the understanding established in *Example 1*, a common misconception is exposed with the comparison of $\sqrt{10+8}$

$\sqrt{3\times 6}$ $\sqrt{10+8}$ $\sqrt{2\times 9}$ $\sqrt{2}\times\sqrt{9}$ $\sqrt{10}+\sqrt{8}$ $\sqrt{3}\times\sqrt{6}$ *Explain your choices.*	and $\sqrt{10}+\sqrt{8}$. Students need to be able to articulate why these are not the same, and that the latter is not equivalent to $\sqrt{18}$.
Example 3: a) *Calculate* $\frac{\sqrt{a}}{\sqrt{b}}$ *and* $\sqrt{\frac{a}{b}}$ *when* $a = 36$ *and* $b = 9$. b) *What do you notice?* c) *Will this always be true for any value of* a *and* b*? Why or why not?*	One of the important understandings underpinning this key idea is which **representations** are equivalent. *Example 3*, like *Example 1*, helps students explore equivalent calculations. This time, the focus is on division with surds. It is essential that students do not only focus on multiplication when looking at multiplicative structures.
Use factorisation in order to rewrite expressions of the form $\sqrt{a}$ into expressions of the form $b\sqrt{c}$ *Example 4:* *Fill in the gaps:* a) $\sqrt{\_}\times\sqrt{5} = 2\sqrt{5}$ b) $\sqrt{\_}\times\sqrt{5} = 4\sqrt{5}$ c) $\sqrt{\_}\times\sqrt{5} = 5\sqrt{5}$ d) $\sqrt{\_\times 5} = 6\sqrt{5}$ e) $\sqrt{\_} = 3\sqrt{5}$ f) $\sqrt{\_} = 10\sqrt{5}$	The **variation** in *Example 4* draws on the understanding developed in *Example 1.* Here, the focus is on preparing students to use the square factors of a number, a, to rewrite the surd $\sqrt{a}$ in the form $b\sqrt{c}$. As the questions progress, students move from writing a multiplier within a product to writing a single value. To supports students' **deepening** understanding of manipulating expressions in surd form, connections should be made to their work on factorising earlier in Key Stage 3. You might find it helpful to revisit the key ideas that explore the distributive law, such as 1.4.3.1 (which is exemplified in the Key Stage 3 PD materials) and 1.4.3.2.
Example 5: *Austin, Ella and Holly are trying to simplify* $\sqrt{48}$. *Austin writes* $\sqrt{48} = \sqrt{12\times 4}$. *Ella writes* $\sqrt{48} = \sqrt{16\times 3}$. *Holly writes* $\sqrt{48} = \sqrt{8\times 6}$. *Who has made the 'best' start? Explain your answer.*	In this example, the **language** of 'best' is used to compare the three different approaches. You may need to unpick with students what is meant by 'best' in this context. Students should recognise that they need to identify any square factors of the number in the root.

Example 6: *Which of these expressions is* ***not*** *equivalent to* $3\sqrt{2}$*?* $\sqrt{3\times 6}$ $\sqrt{\frac{36}{2}}$ $\sqrt{18}$ $\sqrt{10+8}$ $\sqrt{4}\times\sqrt{9}$ $\sqrt{2}\times\sqrt{9}$ $\sqrt{3}\times\sqrt{6}$ $\sqrt{10}+\sqrt{8}$ $\sqrt{20}-\sqrt{2}$ *Explain your choices.*	*Example 6* draws upon the same starting point as *Example 2* and is again concerned with different **representations** of the same number. This time the focus is upon the simplified form of $\sqrt{18}$. Students who understand that $3\sqrt{2}$ is simply another way of writing $\sqrt{18}$ should identify that any surd equivalent to $\sqrt{18}$ will also be equivalent to $3\sqrt{2}$.
Example 7: *Square each of these expressions.* $\sqrt{7}$ $2\sqrt{7}$ $3\sqrt{7}$ $10\sqrt{7}$ $a\sqrt{7}$	This might seem like a superficial set of questions, although students may come unstuck squaring both a surd and non-surd within the same term. **Deepening** students' understanding might come from pausing after each question to ask, *'So what is the square root of the answer?'* This should draw attention to the fact that the square root should, necessarily, return you to the original expression. Therefore if, for example, $(2\sqrt{7})^2 = 4\times 7 = 28$, then $\sqrt{28}$ must equal $2\sqrt{7}$. This offers another way to reinforce that surds can be manipulated and simplified while maintaining their value.
Example 8: *Which of these expressions are integer multiples of* $\sqrt{5}$*?* $\sqrt{10}$ $\sqrt{45}$ $\sqrt{75}$ $\sqrt{30}$ $\sqrt{500}$	The **language** in this question should be familiar, as both 'integer' and 'multiple' are terms first introduced in Key Stage 2. However, students may not be expecting to see these words in an example such as this. Being able to interpret familiar terms in unfamiliar contexts is a key element of a deep and connected understanding. Ensure they understand they are looking to be able to rewrite each expression in the form $a\sqrt{5}$, where a is an integer. The **variation** in this set of examples draws attention to the conditions that are needed for an expression to be an integer multiple of 5: namely, that the number in the root can be rewritten as a product where one factor is 5 and the other is a square number. The three that are not integer multiples should prompt useful discussion. For example, students may initially find $\sqrt{75}$ to be misleading: as 25 is a factor, then it is a multiple of 5 rather than $\sqrt{5}$.

Appreciate that, when a number of expressions are all multiples of the same surd, they can be added (or subtracted) *Example 9:* *Simplify by collecting like terms.* *a)* $a+a+a+a-a+a-a+a$ *b)* $2a+3a-5a+\frac{3}{2}a-6a+0.5a$ *c)* $3a^2-5a^3+4a^2+a^3$ *d)* $\sqrt{5}+\sqrt{5}+\sqrt{5}+\sqrt{5}-\sqrt{5}+\sqrt{5}-\sqrt{5}$ *e)* $4\sqrt{5}-5\sqrt{5}+\sqrt{5}-3\sqrt{5}+2\sqrt{5}$ *f)* $3\sqrt{5}-5\sqrt{3}+4\sqrt{5}+\sqrt{3}$	*Example 9* has its foundations in Key Stage 2 with the concept of unitising and links to working with fractions with the same denominator and collecting like terms in Key Stage 3. Consider whether it is necessary to include this step for **deepening** students' understanding of surds as terms, or whether they are already happy to manipulate surds in the same way they use algebraic unknowns. The **variation** here is such that parts *d* to *f* mirror parts *a* to *c*, so you can look at the expressions in pairs to determine what is the same and what is different.
Example 10: *Derren says that some of the questions below are possible. What might he mean?* *a)* $\sqrt{5}+\sqrt{20}$ *b)* $\sqrt{16}+\sqrt{8}$ *c)* $\sqrt{5}-\sqrt{50}$ *d)* $\sqrt{16}+\sqrt{36}$ *e)* $\sqrt{5}+\sqrt{29}$	The **language** here is ambiguous, to promote discussion about what 'possible' really means in this context. Only part d has an integer answer; only part *a* can be written as a single surd'; and the simplest form of part *e* is already given. Ask students to think of a better way of phrasing the question, perhaps by grouping the calculations into categories such as 'can/cannot be simplified'. This should encourage them to move away from thinking about surds as calculations to be completed, and towards manipulating them as terms
Example 11: *Find the sum of these groups of values.* *a)* $\sqrt{12}$, $\sqrt{27}$, $\sqrt{48}$ *b)* $\sqrt{12}$, $\sqrt{20}$, $\sqrt{32}$ *c)* $\sqrt{10}$, $\sqrt{45}$, $\sqrt{75}$, $\sqrt{30}$, $\sqrt{500}$	The **variation** between parts a and b draws attention to when surds will and will not be like terms. The numbers 12, 27 and 48 all share a factor of 3 (which is not square) and have one other square factor, so $\sqrt{3}$ will remain as a common term. The numbers 12, 20 and 32 all share a factor of 4 (which is square), and so the surd factors that remain will all be different. The important concept to draw out is that the unit has to be the same in order to add or subtract the terms.
Confidently manipulate expressions with surds. *Example 12:* *a)* *Find the area and perimeter of this rectangle.*  *b)* *Find the area and perimeter of this triangle.* 	The tasks in *Example 12* are more involved and offer some ideas for **deepening** students' understanding by connecting their work on surds to already established contexts.

7.2.2.8 Understand and use the technique of rationalising the denominator to transform a fraction to an equivalent fraction

Common difficulties and misconceptions
It is important for students to be accurate in the use of this technique, but this is not alone enough for them to be truly fluent. In addition to being able to perform the technique, they must also: • Understand why it works. • Be familiar with using it in a wide range of instances, for expressions of the form $\sqrt{a}$, $b\sqrt{a}$, $a \pm b\sqrt{c}$ and $a\sqrt{b} \pm c\sqrt{d}$. • Crucially, understand why this technique is useful (i.e., when needing to add or subtract expressions) and choose the appropriate multipliers which enable this to be possible. For expressions of the form, $a \pm b\sqrt{c}$ or $a\sqrt{b} \pm c\sqrt{d}$, students must be fluent with the use of the difference of two squares, and understand that, when a and b are square roots, the transformation of $(a+b)(a-b)$ into $a^2 - b^2$ results in those square roots being eliminated.

Students need to	**Guidance, discussion points and prompts**
Understand the difference between rational and irrational numbers *Example 1:* a) *Which of these numbers is rational?* $\frac{4}{11}$ $\quad$ $\frac{4}{10}$ $0.\dot{3}$ $\quad$ 0.1425 $\sqrt{2}$ $\quad$ $\sqrt{4}$ $\frac{22}{7}$ $\quad$ π b) a *and* b *are integer values. Which of these expressions has a rational denominator?* $\frac{a^2}{a^2+3}$ $\quad$ $\frac{8}{\sqrt{5}+2}$ $\frac{b}{13}$ $\quad$ $\frac{1}{\pi}$ $\frac{\sqrt{3}}{2}$ $\quad$ $\frac{1}{\sqrt{2}}$ $\frac{a}{b}$ $\quad$ $\frac{a}{\sqrt{b}}$	*Example 1* is primarily concerned with establishing understanding of the **language** of 'rational' and 'irrational', before students work more deeply rationalising the denominator. The **variation** here is designed to expose some common misconceptions; you may find it helpful to compare pairs of similar questions. For example, in part a, some students might mistakenly think that $\frac{4}{11}$ is irrational because it doesn't simplify like $\frac{4}{10}$ does; or that $0.\dot{3}$ is irrational because it doesn't terminate like 0.1425 does. The number $\frac{22}{7}$ is a commonly used approximation of π, which might lead to the misconception that it is also irrational. Clarify that 'rational' refers to any number that can be written as a fraction with integer values for both the numerator and denominator. Part *b* focuses on rational denominators and, again, uses pairs of similar examples for **deepening** understanding. The final two might be particularly illuminating: can students suggest values for b that would mean **both** are rational? How about just one? Or neither?

Appreciate that when an irrational number is multiplied by an appropriate term, the resulting number can be rational *Example 2:* *Suggest values for a to e.* $\sqrt{3} \times a = 3$ $\sqrt{2} \times b = 2$ $\sqrt{3} \times c = \sqrt{6}$ $\sqrt{3} \times d = 2\sqrt{3}$ $\sqrt{2} \times e = 4$	Before considering the more complex calculations required for rationalising the denominator in algebraic fractions, it is worth **deepening** understanding of multiplying surds by different terms. Students' approaches to finding a value for e might be particularly illuminating; they may find it easier to work in index form for this. After completing this example, explore the **language** of 'rationalised' as you can identify which of the products has resulted in a number that has been rationalised.
Example 3: *Simplify the following:* *a)* $(\sqrt{3} + \sqrt{2})\sqrt{2}$ *b)* $(\sqrt{3} + \sqrt{2})\sqrt{3}$ *c)* $(\sqrt{3} + \sqrt{2})(\sqrt{3} + \sqrt{2})$ *d)* $(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2})$ e) *Which of the answers is rational?* *f) What are the special features of the terms in the second bracket that have enabled this to happen?*	As with *Example 2,* focus on the **language** of rationalising and which parts of this question are fully rationalised. Discuss which terms are rationalised each time; the 'odd one out' is the last part of the question as it is the only one where both the $\sqrt{3}$ and $\sqrt{2}$ terms have been rationalised. *Example 3* **deepens** understanding of the structures required to rationalise an expression of more than one term. Students need to make connections with previous work on the difference of two squares, likely to have been started but not fully consolidated in Key Stage 3 (see Key Stage 3 PD materials, '*1.4 Simplifying and manipulating expressions, equations and formulae',* KSU 1.4.4). Students should be able to explain why, in part **c**, both $\sqrt{3}$ and $\sqrt{2}$ are rationalised but the new irrational terms of $2\sqrt{3}\sqrt{2}$ is introduced, and also to identify what is different about part *d* that enables the result to be fully rational.
Appreciate that fractions involving surds in the denominator can be expressed as equivalent fractions where the denominator is rational *Example 4:* *What is the value of p in this equation?* $\frac{8}{\sqrt{3}} \times \frac{p}{p} = \frac{8p}{3}$	As with *Example 2*, here we are **deepening** understanding of rationalisation before moving on to more complex expressions. Students tempted to attempt to rearrange and solve the equation using formal methods are less likely to be successful. Encourage them to think about how the numerators and denominators each change. Key to *Example 4* is recognising that, once multiplied by p, the denominator of $\sqrt{3}$ has been rationalised to 3. Therefore p must have a value of $\sqrt{3}$.
Example 5: *Multiply the fraction* $\frac{1}{\sqrt{3}+\sqrt{2}}$ *by* $\frac{p}{p}$ *where:* *a)* $p = \sqrt{2}$ *b)* $p = \sqrt{3}$ *c)* $p = \sqrt{3} + \sqrt{2}$ *d)* $p = \sqrt{3} - \sqrt{2}$	The **variation** here helps students attend to what terms are required to fully rationalise the denominator. As with *Example 3*, they should notice that the values for p in parts *a* and *b* only rationalise one value, while the value in part *c* generates an additional irrational term. See the guidance for *Example 3* for more ideas about what to draw out.

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e) Which of your products has a denominator that is rational?

f) Which of your answers to parts a to d have the greatest value?

g) Now, repeat a) to d) with the fraction $\frac{1}{\sqrt{3}-\sqrt{2}}$. What do you notice?

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Part *f* of *Example 5* ensures that students realise that since $\frac{p}{p} = 1$, multiplying by $\frac{p}{p}$ results in an equivalent fraction each time, so all the answers are equal in value.

Note the connection between the values used for *Examples 3* and *5*. How might you exploit this connection? Refer to the previous task after some time had passed, or complete the tasks in sequence? What are the benefits and limitations of each approach?

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Appreciate that, when fractions have integer denominators, they are easier to calculate with

Example 6:

For which set of fractions, A or B, is it easier to find the sum?

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Students might want to rush in and try to calculate here, but this is not necessary. Instead, focus on **deepening** their understanding of rationalising the denominator by thinking about why it is a helpful step. They should use their existing knowledge of the additive structures for fractions to explain why a common denominator is necessary. This can lead to a discussion of the fact that, when denominators are integers, a common denominator can be found much more easily than if the denominators are irrational.

Discuss with colleagues whether it is worth asking students to find the sum of the expressions for either set. Why or why not? How else might you challenge students within this key idea? Could you, for example, ask them to make up their own examples of where a sum of fractions involving surds is both easy and hard to calculate? What would this reveal about their understanding?

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Set A

$\frac{2}{\sqrt{5}-1}$

$\frac{2}{3-\sqrt{5}}$

$\frac{1}{2+\sqrt{5}}$

$\frac{2}{1+\sqrt{5}}$

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Set B

$\frac{\sqrt{5}-1}{2}$

$\frac{3-\sqrt{5}}{4}$

$\frac{2+\sqrt{5}}{1}$

$\frac{1+\sqrt{5}}{2}$

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7.2.3.3 Appreciate what constitutes the proof of a statement and what is required to disprove it

Common difficulties and misconceptions

Students need to understand the difference between a proof, which is a convincing argument based upon logical deduction, and a demonstration. They may have started to distinguish between the two when working on geometric properties at Key Stage 3 – for example, by looking at different ways to show or prove that the interior angle sum of a triangle is 180°. (See Key Stage 3 PD materials '*6.1 Geometrical properties'*, key idea 6.1.1.2).

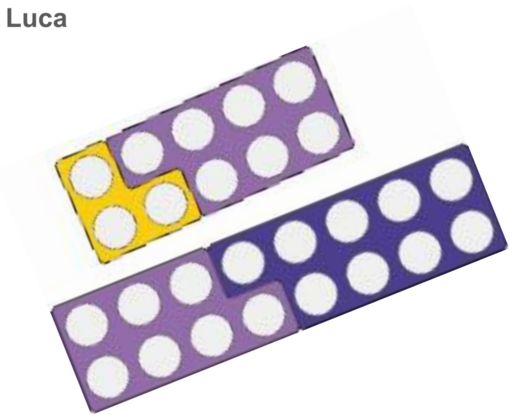
At Key Stage 4, students begin to use algebra to develop logical arguments. It is a big step for students to go from manipulating algebra when responding to questions with instructions (such as 'factorise', 'expand' or 'solve'), to manipulating algebra in order to generalise, demonstrate or prove. There are some common strategies that will support students with their first experiences of algebraic proof, such as creating expressions for numbers with certain properties (for example, $2n$ for an even and $2n \pm 1$ for an odd number). Being able to use factorising to show that a number is even, odd or a particular multiple is also a key skill to develop.

Once students start working on constructing their own algebraic arguments, for example using the structure of 'show that…' or 'prove that…', a common misconception is to use the result as a starting point for their workings. Students need to be aware that this negates the point of the proof, and that the mathematical argument is only valid if their workings reach that conclusion independently of it.

Students need to	Guidance, discussion points and prompts
Understand that generalisations can be made that hold true for any value *Example 1:* *Gaia asks, 'What if we've just not found the one triangle where the angles don't add up to 180°?'* *How would you respond to Gaia?*	*Example 1* focuses on **deepening** understanding by asking students to respond to doubt being cast over a rule that they will have been using in their mathematics learning since Key Stage 2. Students are likely to have seen some kind of demonstration or proof of this rule at some point but may not immediately remember it. The intention is not to check their recall, but to see how they respond to a mathematical certainty being challenged. There is scope here for discussion about mathematical certainty and the difference between trying out some examples and proving beyond all doubt.
Example 2: *Find the next pair of consecutive primes after 2 and 3.*	*Example 2* asks a question with no indication that it is impossible to find subsequent values; listen for the **language** that students use to respond. They should realise that they will not be able to find another pair of consecutive primes. The intention is to draw attention to the fact that there are some mathematical truths that students can state using what they already know, and to see how confident students are in reasoning around them. Consider how this question is framed. Below are some alternative ways to use the same problem. Discuss with colleagues what is the same and what is different about these questions. Is one structure more likely to elicit reasoning than another? • Faisal is asked to find the next pair of consecutive primes after 2 and 3. How should he respond? • Why is it not possible to find another pair of consecutive primes after 2 and 3?

	• Convince me that there are no more consecutive primes after 2 and 3.
Example 3: *a) How can I check if a number is a multiple of 3? Why does this rule work?* *b) How can I check if a number is a multiple of 6? Why does this rule work?*	As with *Example 2*, in *Example 3* we are interested in the **language** students use to explain some mathematical truths. Listen for students who are rooted in particular examples and prompt them to think more generally. Can they explain why this is the case for **all** multiples?
Appreciate the difference between demonstration and proof *Example 4:* *Three students try to prove that the sum of two odd numbers is always even.* *Whose proof do you find most convincing? Why? Could you create a more convincing proof?*	Here we explore different **representations** of the same general principle. The intention is that students' discussion can be focussed on what makes one representation more convincing, or more generalisable, than another. This supports their nascent understanding of proof. Students should, for example, recognise that the first two pictures only show particular cases (albeit several of them, in the case of picture two) rather than a general rule.

Luca



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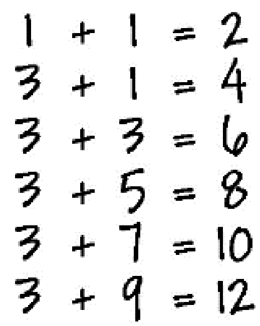
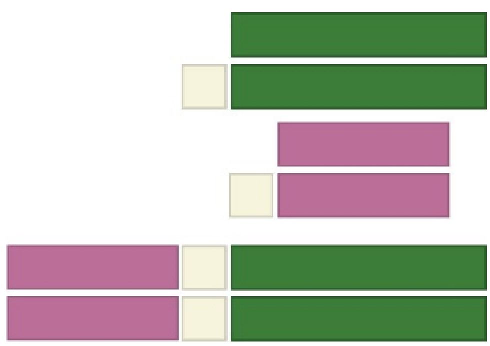


Priya



Example 5:

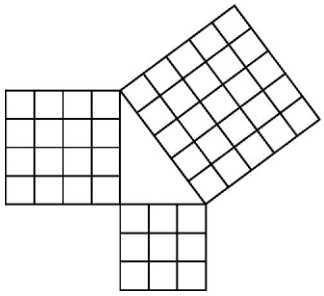
*Three teachers explain Pythagoras' theorem in different ways, below. Whose method do you find most convincing? Whose method **proves** the theorem?*

Here, we explore the **language** of 'proof' and what it means 'to prove' something. Students may have seen different ways to model Pythagoras' theorem before, likely as part of a process to convince them of its validity. They should appreciate that, convincing as it might be for Mr Perry to generate a whole-class set of values that satisfy the rule, this does not constitute a proof unless the examples cover every possible case. Similarly, though it is possible to mathematically prove Ms Hawke's dissection works for all triangles, the process of cutting and rearranging the shapes is a demonstration not a proof.

Consider **deepening** students' understanding of Ms Akhtar's proof by only revealing the instruction, and not the algebra itself. Students should be able to form and manipulate these expressions using their understanding from Key Stage 3.

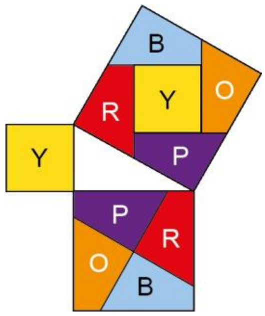
***Mr Perry** asks students to draw right-angled triangles and the square of each side in their books.*

The class draw lots of different triangles, with their squares, and record their results. They notice that the number of squares for the two shorter sides is always the same as the number of squares for the longer side.



***Ms Hawke** draws squares using the lengths of each side.*

When she cuts up the smaller two squares, she shows she can rearrange the pieces to fit exactly into the larger square.

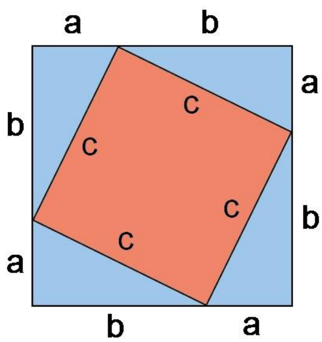


***Ms Akhtar** draws four identical right-angled triangles with lengths a and b and arranges them in a big square so their hypotenuses, c, create a smaller, tilted square.*

She reasons that she could work out the area of the whole larger square using $(a+b)(a+b)=a^2+2ab+b^2$.

She notices she could also work it out another way, by adding up the areas of the four triangles and the smaller square: $4\times\frac{1}{2}ab+c^2=ab+c^2$.

Since both of these are expressions for the same area, they must be equal, so she writes:



$$a^2+\cancel{2ab}+b^2=\cancel{2ab}+c^2$$

$-2ab]$ $\quad$ $[-2ab$

$$a^2+b^2=c^2$$

Understand that counter examples can be used to disprove a conjecture

Example 6:

Rowan notices that the numbers either side of 6 are both prime.

He checks the numbers either side of 12, and notices these are both prime too.

He tries a much higher number, 30, and notices the same thing.

He says, 'Every multiple of 6 has a prime number either side.'

Do you agree with him? Why or why not?

Examples 6 and *7* support students' **deepening** understanding of proof and what is required to disprove a statement. In both cases, a pattern established for the first few numbers of a sequence does not hold for the rest of the sequence. Students need to be aware that a few or even many examples are not sufficient to prove a conjecture to be true, whereas a single counter-example will disprove a conjecture.

Students may be interested in some examples of other unproven problems in mathematics. Possibly the most accessible for students at Key Stage 4 is the Collatz conjecture, which is unproven even though it has been shown to be true for all starting numbers up to 2^{68}.

Example 7:

Bryony is investigating factorials.

She writes the first four factorials.

$1!$	1	1
$2!$	2×1	2
$3!$	$3 \times 2 \times 1$	6
$4!$	$4 \times 3 \times 2 \times 1$	24

She says, 'I've noticed something interesting: 1 has 1 factor, 2 has 2 factors, 6 has 4 factors and 24 has 8 factors. So, the number of factors doubles every time. I predict that $5!$ *will have 16 factors and* $6!$ *will have 32 factors.'*

Do you agree with her? Why or why not?

Example 7 could be used as an alternative *to Example 6*. What are the benefits and challenges of using this example, with less familiar maths, instead of the more familiar maths of *Example 6*?

Use algebra to express numbers with specific properties

Example 8:

Abi, Bashaar and Cathy each have a pile of paper clips.

Abi tips all her paper clips into one box.

Bashaar splits his paper clips equally between two boxes.

Cathy splits her paper clips equally between three boxes.

a) Who can you be ***certain*** *has an even number of paper clips?*

Example 8 offers a **representation** for unknown numbers to expose students' thinking around odd and even numbers. It draws on the ideas needed to understand why the expression $2n$ always represents an even number, and that $2n + 1$ an odd number. Students should be able to articulate why we only know for certain, in part *a*, that Bashaar has an even number of paperclips – perhaps by describing each paper clip in one box having a matching pair with another. Similarly, for part *b*, the same logic can be applied to explain how we know Bashaar doesn't have an even number, whereas Abi will have and Cathy might have.

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b) How would your answers change if each person had one paper clip left over?

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Example 9:

If n represents any positive integer, how could you show:

a) the next number

b) another consecutive number

c) an even number

d) another even number

e) an odd number

f) another odd number

g) a multiple of 7

h) another multiple of 7

i) 3 less than a multiple of 5

j) another number 3 less than a multiple of 5?

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Example 9 explores different algebraic **representations** of generalised numbers. Students need to apply their understanding of properties of number (for example, that even numbers are multiples of 2) to the unknown, n.

Throughout, the **variation** is designed so that students need to think of more than one way to express the same property. It is helpful to discuss whether this really is a different number, or a different way of expressing a number with the same properties.

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Example 10:

Sort the expressions into the table.

n $\quad n+1 \quad n+2 \quad 2$

$7n \quad 7n+1 \quad 7(n+1) \quad 8n$

$8n+2 \quad 8n-1 \quad 8(n-1) \quad 8n+n$

Always odd	***Could be odd or even***	***Always even***

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Example 10 **deepens** understanding of algebraic expressions by asking students to sort them into different categories. This essential understanding underpins the use of algebra to prove a conjecture. Students need to be aware of the role of the coefficient in determining many proofs, so that they can identify what it is possible to conclude from these expressions. Ask questions to generalise from this experience to support this understanding.

- *'If a coefficient is odd, does this mean that the expression is an odd number?'*
- *'If a coefficient is even, does this mean that the expression is an even number?'*
- '*If I add an odd number to an expression, does this mean that the expression is odd?'*

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Interpret an algebraic statement to draw a conclusion

Example 11:

Look at these numbers, expressed algebraically, where x is an integer.

$10(x-4)$	$3(x+1)$	$3x+1$
$x+5$	$5x+4$	$6x+3$

a) Which of these is:

i. always a multiple of 5

ii. sometimes a multiple of 5

iii. always a multiple of 3

iv. sometimes a multiple of 3

v. always adjacent to a multiple of 3

vi. always divisible by 2?

*b) Write other statements that you know are true about these numbers. Can you write a statement that is true for **all** the numbers?*

The **variation** in *Example 11* enables students to really examine the structure of the numbers and what properties they reveal. For example, they need to be able to identify how the differences between $3(x+1)$ and $3x+1$ mean that the former is a multiple of 3 and the latter is exactly one more than a multiple of 3. Deliberately, there are no examples where the multiple of 5 is revealed by a multiplier of 5. This means students must think about which part of an expression is indicative of a number being a multiple, and what conditions the rest of the expression needs to meet.

Draw attention to the fact that when an expression always holds true for a particular statement, it can be used in algebraic proof to represent numbers with that attribute.

Example 12:

Sam is asked to show that, for any three consecutive integers, the difference between the squares of the first and last numbers is four times the middle number.

a) He starts by writing expressions for three consecutive integers. What might these be?

b) Sam then writes his first and last expression in brackets and multiplies them. What might his workings look like?

c) Sam then compares his new expression to the expression he wrote for the middle number. How might he do this?

Example 12 offers a structure for supporting students through a problem to **deepen** their understanding of what is required to ‘show’ something algebraically. Students may offer different solutions to part a. It is useful to compare them and decide which are valid expressions for three consecutive integers.

Then it is helpful to consider efficacy. For example, is there a benefit in using $n-1$, n and $n+1$ over n, $n+1$ and $n+2$?

Modelling parts *b* and *c* with different sets of three expressions to decide which is the most efficient will help students to build their skills.

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Example 13:

Sienna is asked to prove that the difference between the squares of any two consecutive odd numbers is always a multiple of 8.

She writes: $(n + 3)^2 - (n + 1)^2$
$= n^2 + 6n + 9 - n^2 - 2n - 1$
$= 4n + 8$

a) *Why has her proof not worked?*

b) *Improve her proof.*

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In *Example 13*, students need to consider the algebraic **representations** they use for a number and what detail must be included. They may be tempted to look for a mistake in Sienna's algebraic manipulation, but the issue is with her starting expressions. Can students explain why $n + 3$ and $n + 1$ could be odd or even, and how to ensure that they 'force' the expression to be odd?

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Collaborative planning

Although they may provoke thought if read and worked on individually, the materials are best worked on with others as part of a **collaborative professional development** activity based around planning lessons and sequences of lessons.

If being used in this way, is important to stress that they are not intended as a lesson-by-lesson scheme of work. In particular, there is no suggestion that each key idea represents a lesson. Rather, the fine-grained distinctions offered in the key ideas are intended to help you think about the learning journey, irrespective of the number of lessons taught. Not all key ideas are of equal weight. The amount of classroom time required for them to be mastered will vary. Each step is a noteworthy contribution to the statement of knowledge, skills and understanding with which it is associated.

Some of the key ideas have been extensively exemplified in the guidance documents. These exemplifications are provided so that you can use them directly in your own teaching but also so that you can critique, modify and add to them as part of any collaborative planning that you do as a department. The exemplification is intended to be a starting point to catalyse further thought rather than a finished 'product'.

A number of different scenarios are possible when using the materials. You could:

- Consider a collection of key ideas within a core concept and how the teaching of these translates into lessons. Discuss what range of examples you will want to include within each lesson to ensure that enough attention is paid to each step, but also that the connections between them and the overall concepts binding them are not lost.
- Choose a topic you are going to teach and discuss with colleagues the suggested examples and guidance. Then plan a lesson or sequence of lessons together.
- Look at a section of your scheme of work that you wish to develop and use the materials to help you to re-draft it.
- Try some of the examples together in a departmental meeting. Discuss the guidance and use the PD prompts where they are given to support your own professional development.
- Take a key idea that is not exemplified and plan your own examples and guidance using the template available at Resources for teachers using the mastery materials | NCETM.

Remember, the intention of these PD materials is to provoke thought and raise questions rather than to offer a set of instructions.

Solutions

Solutions for all the examples from *Theme 7 Using and applying numerical structure* can be found at

https://www.ncetm.org.uk/media/y01pnrgg/ncetm_ks4_cc_7_solutions.pdf