

## Welcome to Issue 132 of the Secondary and FE Magazine

Can it be nearly the end of April already, with the exam season almost upon us? Despite the importance of this term for all secondary teachers, we hope you'll have time to delve into the inner reaches of our monthly magazine. Let us know what you think, by email to [info@ncetm.org.uk](mailto:info@ncetm.org.uk) or on Twitter [@NCETMsecondary](https://twitter.com/NCETMsecondary).

### Contents

#### [Heads Up](#)

Here you will find a checklist of some of the recent, or still current, mathematical events featured in the news, by the media or on the internet: if you want a "heads up" on what to read, watch or do in the next couple of weeks or so, it's here. If you ever think that our heads haven't been up high enough and we seem to have missed something that's coming soon, do let us know: email [info@ncetm.org.uk](mailto:info@ncetm.org.uk), or via Twitter, [@NCETM](https://twitter.com/NCETM).

#### [Sixth Sense](#)

We eavesdrop on a conversation (thanks to an article from the ATM archives) between a couple of mathematicians about cutting up a toad-in-the-hole to share between three people, but only by making cuts that divide any given piece in half.

#### [From the Library](#)

A look at the research linked to the aim of helping pupils make connections between different areas of mathematics when encountering algebra.

#### [It Stands to Reason](#)

How pictorial representations can help students develop deep understanding of multiplication.

#### [Eyes Down](#)

Thoughts prompted by a student's apparent blind spot when it comes to rounding to significant figures.



## Heads Up



The NCETM has recently published a [new set of webpages](#) to support professional development in the area of multiplicative reasoning in Key Stage 3 classes.



Students who don't get the grade they're after in GCSE maths this summer will now get two chances to resit: once in November this year and once in May or June next year. The technical term for this area is 'legacy resits' and you can read more [here](#).



Is there a link between logarithmic functions and love? A [recent episode](#) of the ever-probing Radio Four programme *More or Less* found a maths teacher who thinks there is, and who wrote a song about it. It's about five minutes into the programme, but why not listen to the whole half hour?



Two items to recommend from our friends at the Further Mathematics Support Programme (FMSP). First their [Spring newsletter](#), and second, a new leaflet on girls' participation in A level maths, which can be downloaded from the [FMSP website](#).

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## Sixth Sense

Can an object be cut precisely into thirds simply by strategic halving? How about using halving to cut it precisely into fifths? In this month's Sixth Sense we reproduce an article from the [ATM](#) which features a not entirely fictitious conversation between Mike Ollerton and Jonny Griffiths in which they discuss a particularly mathematical way to share a toad-in-the-hole!

- Read [Perfect halving, bicimals, and a toad-in-the-hole](#)

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You can find previous *Sixth Sense* features [here](#).



## From the Library

Twenty years on since he published his famous result, Sir Andrew Wiles has been awarded the prestigious Abel prize for his “stunning proof” of Fermat’s Last Theorem. Why such a delay? While the proof put to bed a notorious problem that had been taunting the mathematical community for 300 hundred years, perhaps the main reason for the award is that the results Wiles proved had significance in “opening a new era in number theory”, implications that have taken time to be realised. In particular he proved results that connected two quite different areas of mathematics, number theory and modular forms.

As Alex Bellos explained at the announcement of the prize in Norway, Wiles’s proof demonstrated equivalence between the integer solutions of the equation  $x^n + y^n = z^n$ , the rational coordinates on elliptic curves (curves given by the form  $y^2 = ax^3 + bx + c$ ) and modular forms – highly symmetric mappings related to the geometry of complex numbers. This facility for making connections and shifting between multiple representations of mathematical ideas, so powerfully demonstrated in Wiles’s proof, is built into the aims of the new National Curriculum at Key Stages 3 and 4:

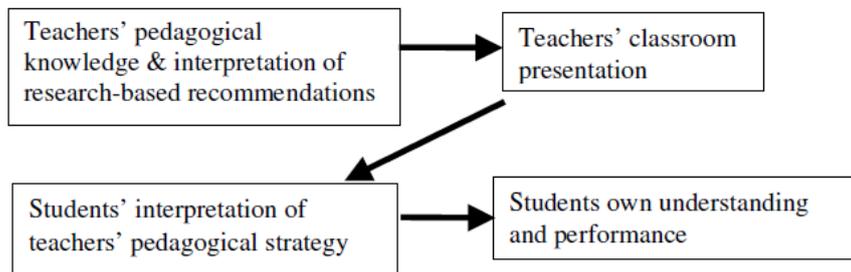
*Mathematics is an interconnected subject in which pupils need to be able to move fluently between representations of mathematical ideas... pupils should make rich connections across mathematical ideas to develop fluency, mathematical reasoning and competence in solving increasingly sophisticated problems*

In particular they need to be able to “move freely between different numerical, algebraic, graphical and diagrammatic representations” (Department for Education, 2014).

Research into multiple representations in teaching and learning algebra has been undertaken since at least the 1980s (p. 12, Kieran, 2006; summary of research pp. 18 – 25). Liz Bills (2001) examined shifts in meanings of literal symbols amongst 16 – 17 year olds to identify “the kind of thinking required to move between different uses” of letters in algebra. Building on previous work (e.g. [Küchemann](#), 1981) she considered questions where letters took on different roles within the same problem. “My study led me to the conclusion that very many of the problems that my students were tackling involved a subtle shift in the role played by the literal symbols; moreover it was this shift which in each case provided the power that made the solutions to these problems ‘standard methods’.” Her first shift considered the question: Solve the simultaneous equations  $x + 2y - 4 = 0$  and  $y = 2x - 2a + b$ . In moving from the perspective of two isolated equations to two related simultaneous equations, the roles of  $x$  and  $y$  change. They move from those of “variable” (“a quantity whose importance is entirely in its relationship with another quantity”) to “unknown-to-be-found” ( $x$  and  $y$  take on the roles of unknowns, whose numerical values are to be found). In a second problem, “What is the equation of a straight line with gradient 3 which passes through the point (2, 8)?”, use of the standard form  $y = mx + c$  involves making a shift for the letter  $c$  from “placeholder-in-a-form” to “unknown-to-be-found”. For after substituting  $x = 2$  and  $y = 8$ ,  $c$  becomes the unknown-to-be-found in the equation  $y = 3x + c$ .

Bills drew the conclusion that the subsequent improvement in student performance indicated that “it may be that drawing their attention to the shifts in meanings involved in some standard problems and their routine solutions would be an effective way of helping students to improve such understanding.”

Amit and Fried (2005), however, found difficulties with drawing students’ attention to multiple representations. In particular they found a disparity “to a great degree” between “teachers’ and students’ interpretations of the meaning and intent of the classroom activity”, represented schematically below:



from Amit and Fried, 2005

In a sequence of 15 lessons on linear equations the teacher wanted “the students to know what representations and the act of representations are all about” and in particular for students to see “equations in a different light.” But to the students, while solving equations carried weight, drawing graphs was just another task, “a redundant exercise”. The students “do not appear to understand them as showing different mutually reinforcing views of the linear equation.” The authors suggest that the absence of mediating elements was a possible contributing factor, that the presence of “connectors” is needed as well as different representations. They conclude that “it may be that we have to challenge a multiple representations approach as a framework to begin with in teaching and think of (it) as a distant goal that may not be achieved until the learner has had considerable experience in kinds of thinking that potentially link representations.”

Focussing on representing processes rather than objects, Davis and McGowen (2002) identified significant success amongst a group of 87 college students who had to revisit functions as part of their “developmental algebra” classes. The research assessed the students’ “flexibility” of thought - which they defined as encompassing Krutetskii’s (1969) “reversibility” (for example, recognising and applying the idea of the inverse of a function), Gray and Tall’s (1994) “proceptual thinking” (see *From the Library* in [Issue 126](#)) as well as “connections between various representations, including tables, graphs and algebraic syntax.” Through use of “function machines” as representations, the pre- and post-test responses as well as the student self-evaluations indicated a significant improvement in understanding of functions as processes and a “dramatic change in flexibility of algebraic thinking.”

Coles (2014), again rather than emphasising concepts, focusses on relationships. He makes a distinction between “absolute” and “relational” representations, terminology he attributes to a suggestion by Tim Rowland. In examining the teaching approaches of Caleb Gattegno and Bob Davis, Coles suggests that it is the relationship between symbols that is important and which provides meaning. The approach is to “set up contexts or structures in which symbols can be introduced with a limited number of dimensions or variations” (following Marton and Booth (1997); Mason (2011)) so that “symbols represent relationships between the objects, or actions on those objects.” Coles illustrates this with the example of Gattegno using Cuisenaire rods to develop the concept of “number” as a relationship to a “unit” (see the video of the [Gattegno’s lesson](#)) and Davis working with negative numbers and developing them through the idea of relationships between numbers. “Symbols soon become meaningful in their connections to each other, and not linked directly to particular objects.”

Enabling students to make connections may be problematic but it is one of the keys to the power and pleasure of mathematics. As Sir Andrew Wiles said of the expanding role played by modular forms - “objects that really come out of geometry” - in number theory: “It’s a very surprising connection and I think it shows something very, very deep in mathematics and the more we study it the more surprised (we become) ... and the more beautiful it seems.”

## Resources

Activities developed by the Standards Unit for Improving Learning in Mathematics (ILIM) provide tasks for linking representations and are available to download at the National STEM online library:

### Mostly Algebra

- A1 Interpreting algebraic expressions
- A7 Interpreting functions, graphs and tables
- A11 Factorising cubics
- A14 Exploring equations in parametric form

### and Mostly Calculus

- C1 Linking the properties and forms of quadratic functions
- C3 Matching functions and derivatives
- C4 Differentiating and integrating fractional and negative powers

While the Singapore Bar Method is often aimed at primary school children, [this explanation](#) of the representation leads to multiplicative problems suitable for Key Stage 3. Also this research by [Spencer and Fielding](#) on using the Singapore Bar for word problems may be of interest.

For exploring understanding of mean average: [An idea for the classroom - multiple representations](#).

Realistic Mathematics Education (RME), described in *From the Library* in [Issue 131](#), uses increasingly abstract “models” to represent and mathematise “realistic” contexts.

## References

Amit, M., & Fried, M. N. (2005). [Multiple representations in 8th grade algebra lessons: Are learners really getting it?](#) In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th PME International Conference*, 2, 57–64.

Bills, L. (2001) Shifts in the meanings of literal symbols. In M. van den Heuvel-Panhuizen (Ed.) [Proceedings of the 25<sup>th</sup> conference of the International Group for the Psychology of Mathematics Education](#) (Vol. 2, pp. 161-168). Utrecht, The Netherlands: PME.

Coles, A. (2014) [Absolute and relational representations: the challenge of Caleb Gattegno and Bob Davis](#) in Barmby, P. (Ed.) *Proceedings of the British Society for Research into Learning Mathematics* 34(1) March 2014

Davis, G.E. and McGowen, M.A., (2002) Function Machines & Flexible Algebraic Thought. [Proceedings of the 26th PME International Conference](#).

Department for Education (2014) [Statutory guidance National curriculum in England: mathematics programmes of study](#)

Gray, E.M. and Tall, D.O. (1994). [Duality, ambiguity, and flexibility: A “proceptual” view of simple arithmetic](#). *Journal for Research in Mathematics Education*, 25 (2), 116-140.



Kieran, C. (2006) Research on the Teaching and Learning of Algebra in Gutierrez, A. and Boero, P. (eds.) *Handbook of Research on the Psychology of Mathematics Education. Past, Present and Future. PME 1979-2006.* Sense Publishers.

Krutetskii, V.A. (1969) An investigation of mathematical abilities in schoolchildren. In Kilpatrick, J. and Wirszup, I. (Eds.) *Soviet Studies in the Psychology of Learning and Teaching Mathematics*, vol I, pp. 5 – 57. Chicago: University of Chicago Press.

Marton, F., & Booth, S. (1997). *Learning and awareness*. Mahwah, NJ: Lawrence Erlbaum Associates.

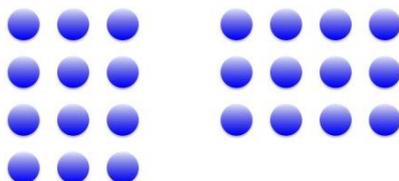
Mason, J. (2011). Explicit and implicit pedagogy: variation theory as a case study. *Proceedings of the British Society for Research into Learning Mathematics*, 31(3), 107-112

You can find previous *From the Library* features [here](#).

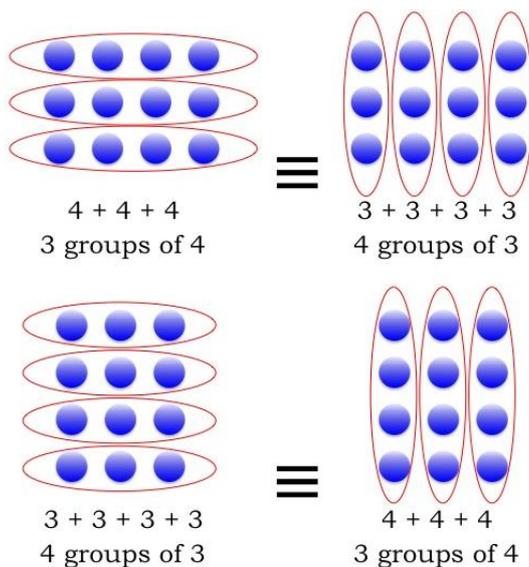


### It Stands to Reason

There has been some discussion among mathematics teachers on Twitter recently about whether the order in which the numbers appear in expressions of products of two numbers should have any significance for pupils, and, if so, what that significance should be. The discussion started with a debate about which of these two images ...



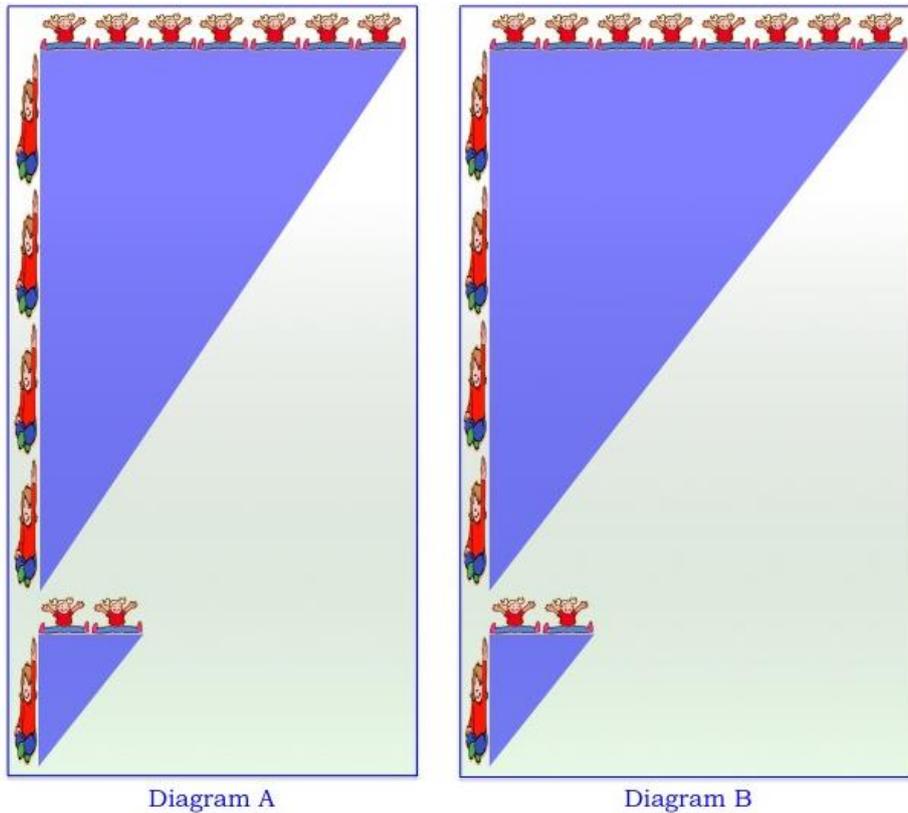
... represents the expression '3 × 4', and which represents '4 × 3'. It was also seen as a debate about which of those two expressions is represented by '3 groups of 4 objects' and which is represented by '4 groups of 3 objects' – which is equivalent to 4 + 4 + 4 and which to 3 + 3 + 3 + 3? Several contributors to the discussion made comments supporting the view that, since ...



... pupils should understand that both '3 × 4' and '4 × 3' can be thought of (pictured) in any of these 8 ways.

Pupils' thinking about multiplication and division is enriched, and their ability to reason multiplicatively is improved, when they can not only call upon repeated-addition images of multiplication (as above) for support but can also use images involving scaling. So we will look at how scaling-images of multiplication can deepen pupils' understanding of multiplication and division, particularly when operating with and on fractions.

Here is a task for the start of a learning session about scaling-images and multiplication ...



***What is the same and what is different about Diagram A and Diagram B?***

Ask students to comment in response to this question, and collect responses on the board without comment.

Encourage pupils to discuss their responses.

Look out for discussion of these samenesses:

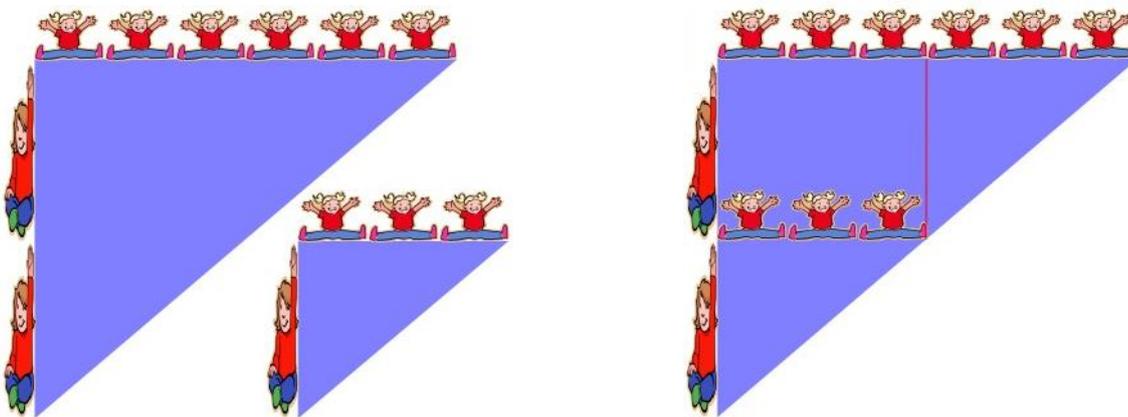
- in both diagrams both triangles are right-angled triangles
- in both diagrams the vertical side-length of the smaller triangle is the height of one 'pointing-upwards child'
- in both diagrams the vertical side-length of the larger triangle is the height of four 'pointing-upwards children'

Look out for discussion of these differences:

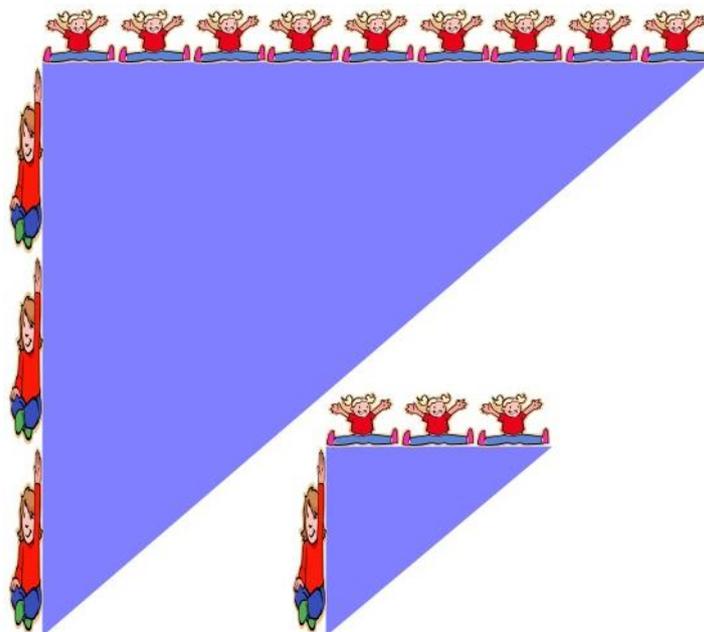
- in Diagram B the larger triangle is a scaled-up version of the smaller triangle,
- in Diagram A the larger triangle is NOT a scaled-up version of the smaller triangle,
- in Diagram B the two triangles are similar
- in Diagram A the two triangles are NOT similar
- in Diagram B the number of 'children with stretched-out legs' on the large triangle is 4 times the number of 'children with stretched-out legs' on the small triangle
- in Diagram A the number of 'children with stretched-out legs' on the large triangle is NOT 4 times

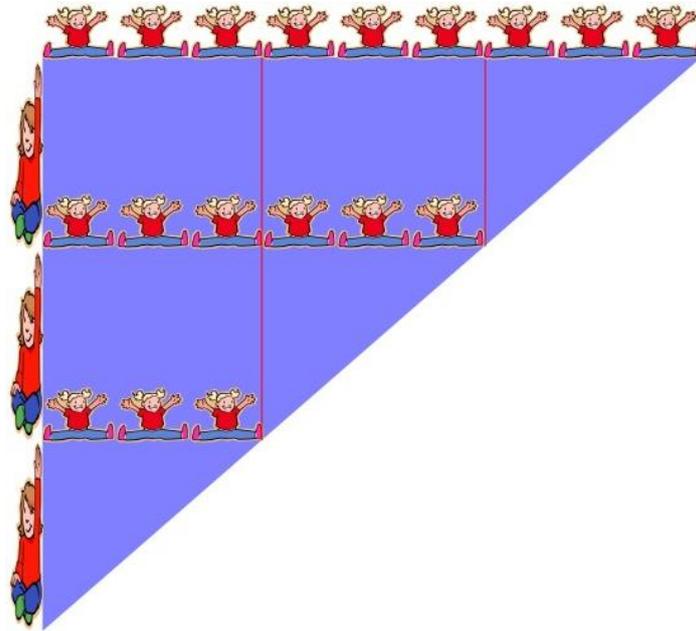
the number of 'children with stretched-out legs' on the small triangle

A right-angled triangle is a useful shape to use in 'scaling-pictures' of multiplication. If it is oriented with the non-hypotenuse sides horizontal and vertical, a number to be multiplied (say  $x$ ) can be represented by the length of the horizontal side. When the triangle is scaled-up (to represent multiplication by a number greater than 1) the length of the horizontal side will be multiplied by the same number as the length of the vertical side is multiplied by. So, if the vertical side-length of the starting-triangle is one-of-something, and the corresponding vertical side-length of the scaled-up triangle is  $n$ -somethings, the length of the horizontal side of the scaled-up triangle represents  $n \times x$ . For example, multiplication of 3 by 2 ( $2 \times 3 = 6$ ) could be represented by scaling-up a right-angled triangle with horizontal side-length 3 units (a 'child with stretched-out legs' is one unit) so that the vertical side-length (the height of a 'pointing-upwards child') is doubled ...



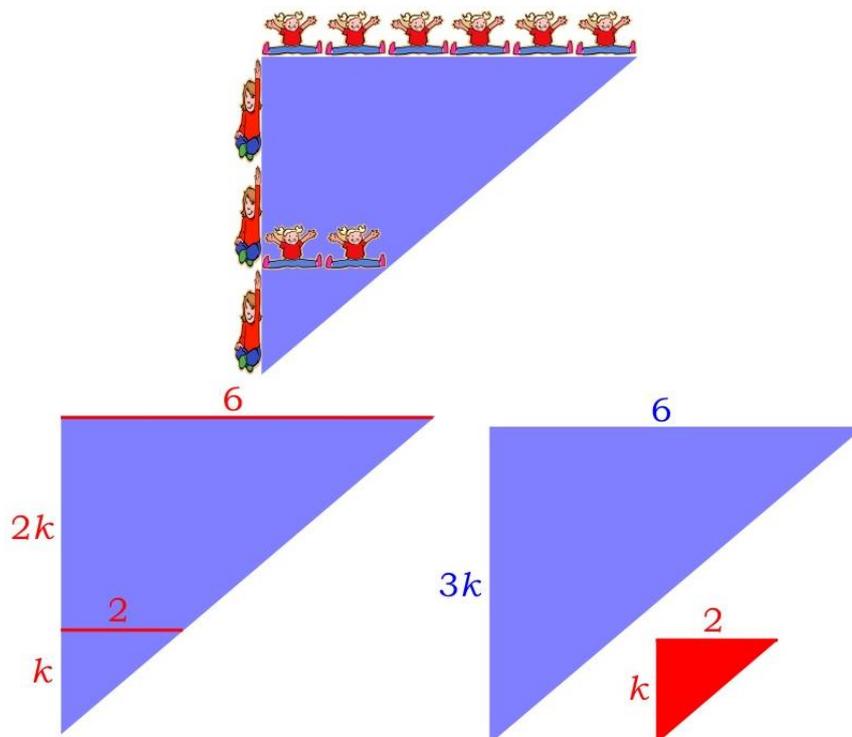
... and  $3 \times 3 = 9$  could be shown by scaling-up the same triangle so that the vertical side-length is tripled:



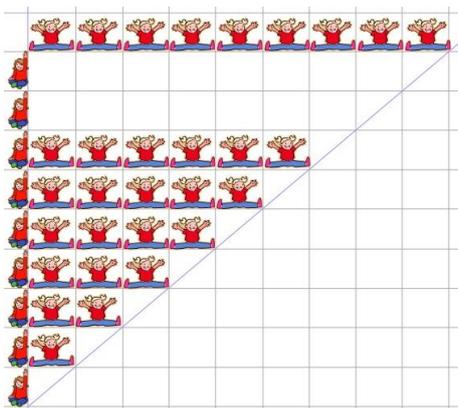


Using one tiny image (in this example a 'child with stretched-out legs') to show each unit on the horizontal side and a different tiny image (in this example a 'pointing-upwards child') to show the vertical side-length of the starting triangle is an effective aid in helping pupils grasp the basic structure of what happens.

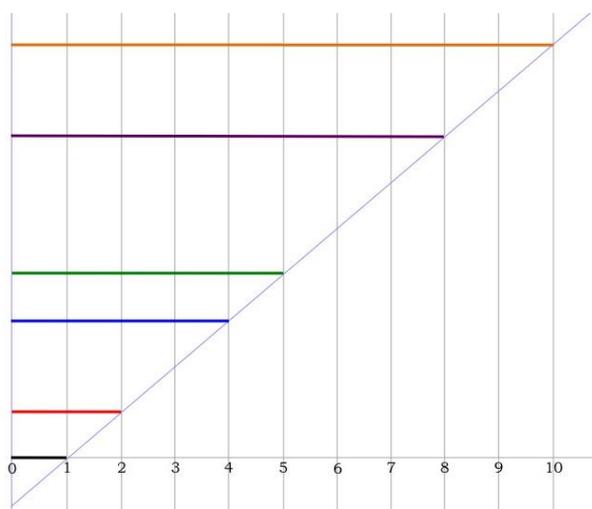
Once pupils understand that whatever the vertical side-length is multiplied by the horizontal-side-length-in-units (the number to be multiplied) is also multiplied by, the 'tiny images' can be dropped, as in this image showing multiplying 2 by 3 ...



Pupils should be prompted (if they don't come-up with it themselves) to see that several different products can be represented in one image. For example, products represented in the next image include  $1 \times 2 = 2$ ,  $1 \times 3 = 3$ , (and so on),  $2 \times 2 = 4$ ,  $2 \times 3 = 6$ ,  $3 \times 2 = 6$ ,  $3 \times 3 = 9$ ,  $1.5 \times 6 = 9$  (vertical side-length scaled-up from  $6k$  to  $9k$ ),  $1.5 \times 4 = 6$  (vertical side-length scaled-up from  $4k$  to  $6k$ ),  $1.5 \times 2 = 3$ , and others that you could challenge some pupils to state.

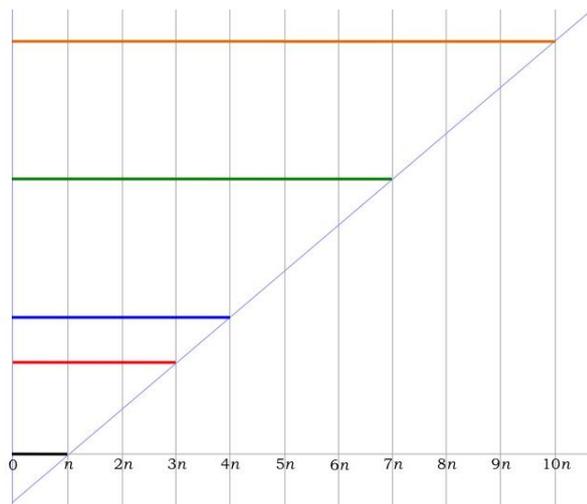
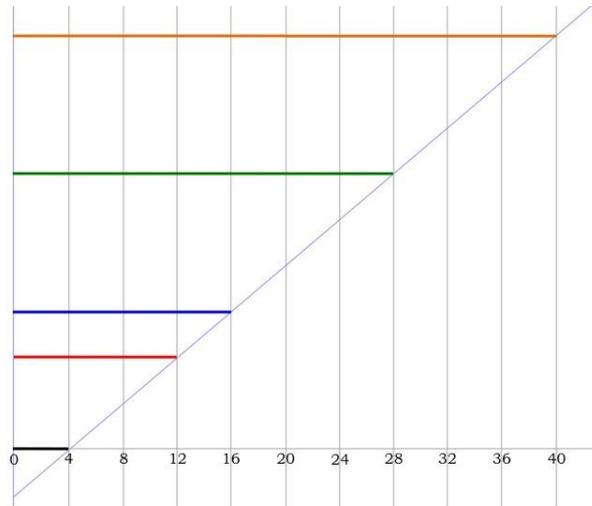


As pupils gain confidence not only the tiny images, but also most of the horizontal grid-lines, can be dropped, as long as the units are shown by equally-spaced vertical grid-lines. For example, you could ask what products this image can represent:

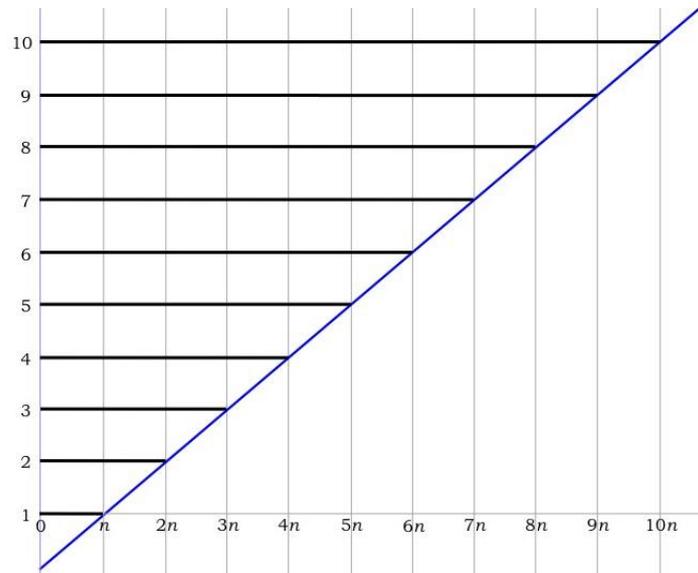


Challenge pupils to explain why the oblique line removes the need to show all the horizontal grid-lines (they should visualise similar right-angled triangles such as those that were previously explicitly shown).

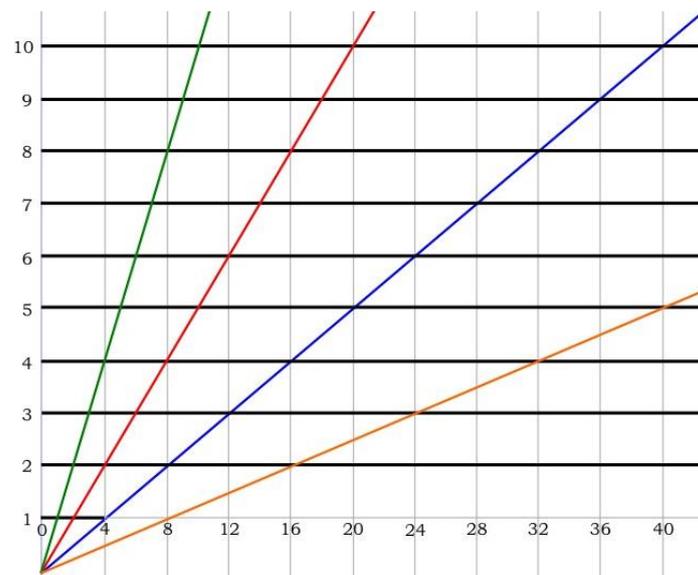
The scope of the image can be increased by marking the vertical lines in multiples of any number, for example in multiples of 4, as shown here:



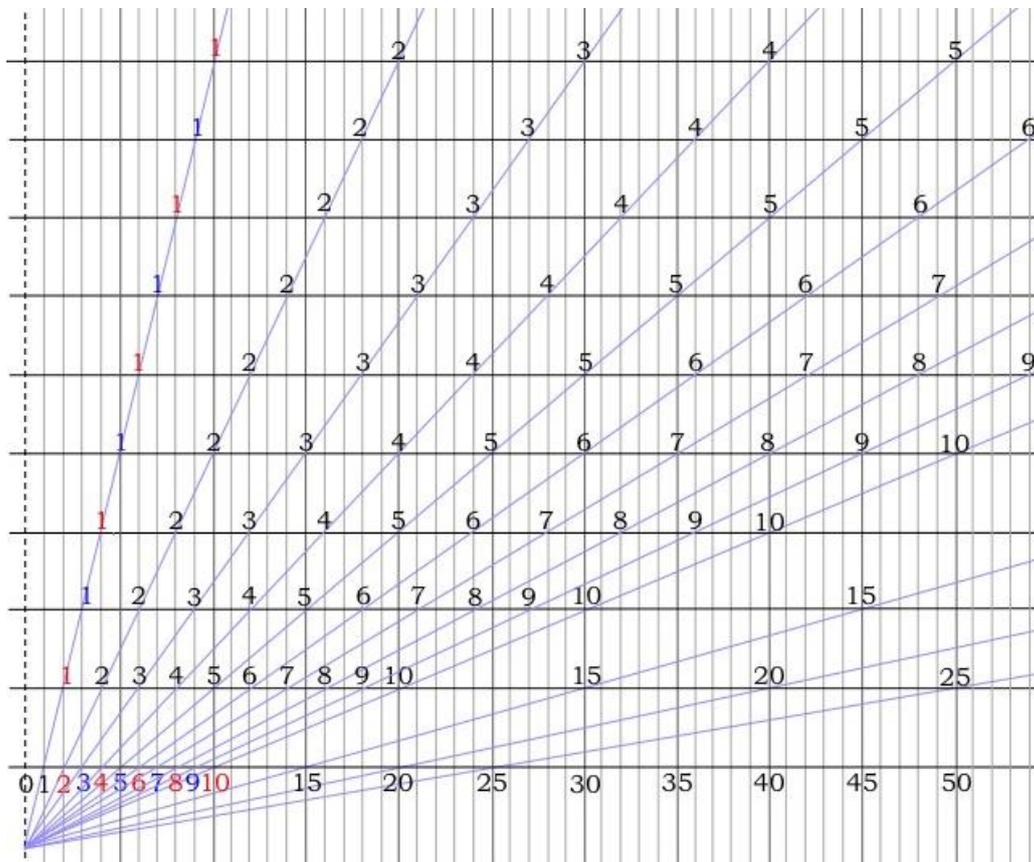
If the equal intervals on the vertical axis are numbered consecutively the image becomes a times-table representation:



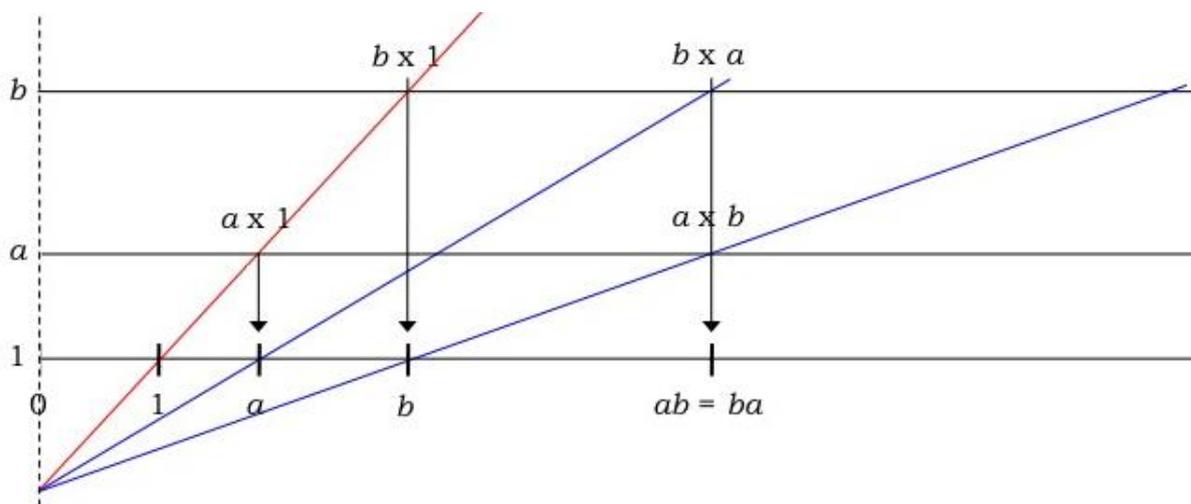
When 'slant-lines' at different angles to the vertical are introduced several times-tables can be represented in one image. For example, in this image ...



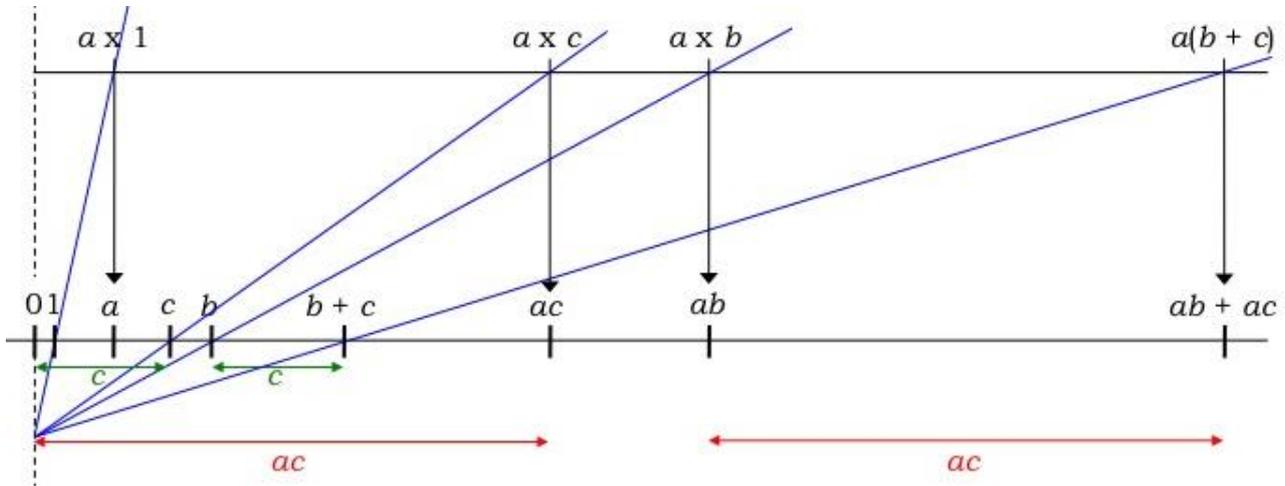
... the red line shows multiples of 2, the blue line shows multiples of 4, and the orange line shows multiples of 8. Challenge pupils to explain how they show these multiples. You could also ask them what the green line shows. The following image gives many more products, which are indicated in new ways. Pupils could discuss the structure of this image and how they can 'read-off' products from it.



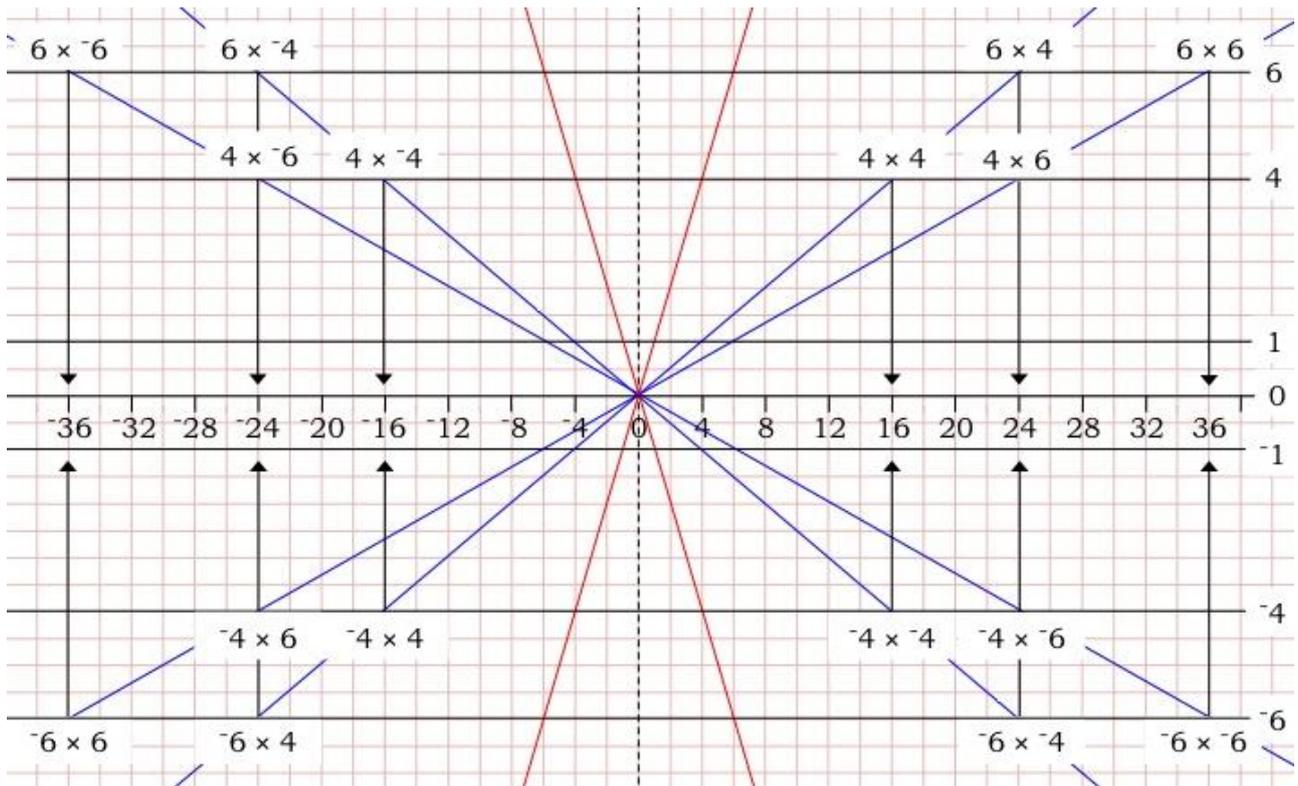
Can pupils see in the above image (or a similar one of their own making) examples of general facts about multiplication? Most pupils should be able to find particular examples of the commutative property of multiplication, as shown generally in this image ...



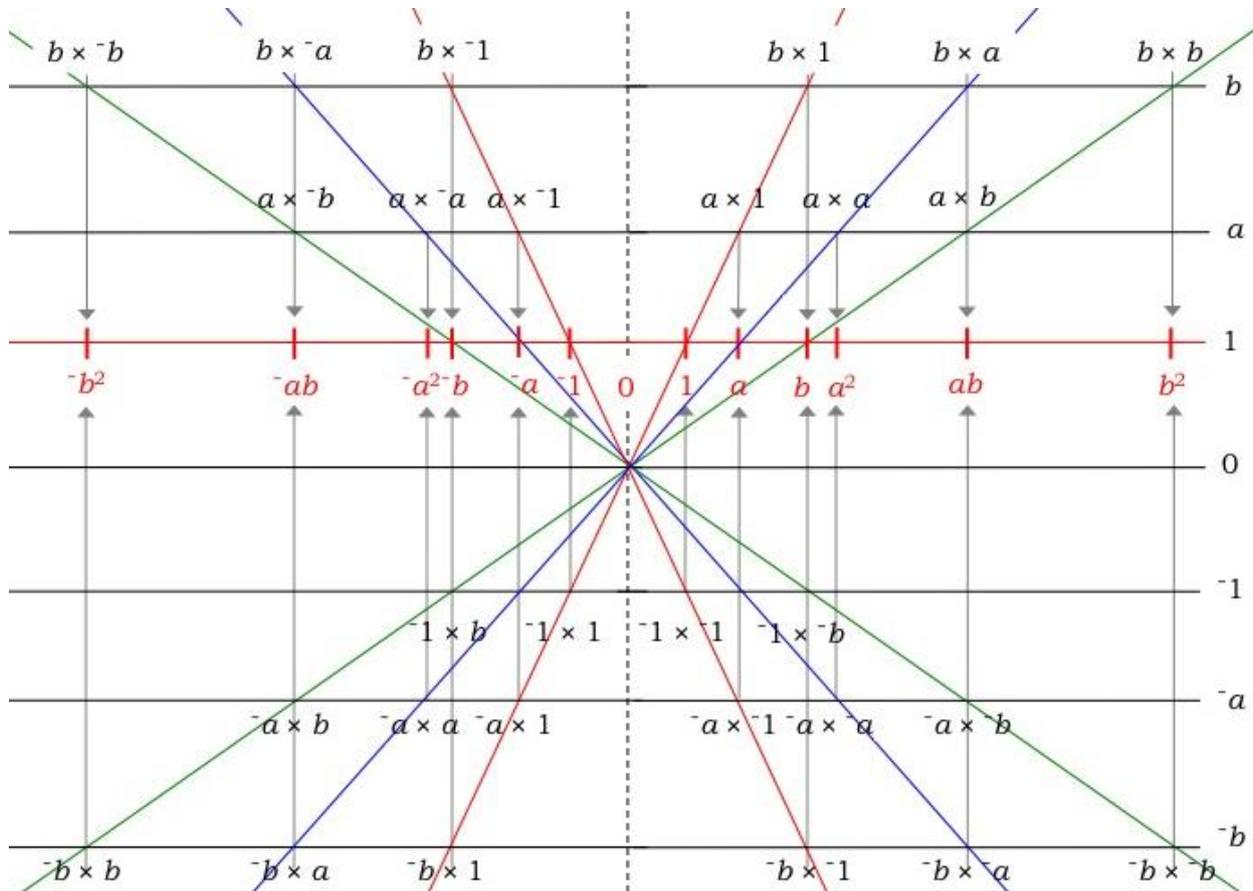
... and some may spot examples of the fact that multiplication is distributive over addition, as indicated generally here:



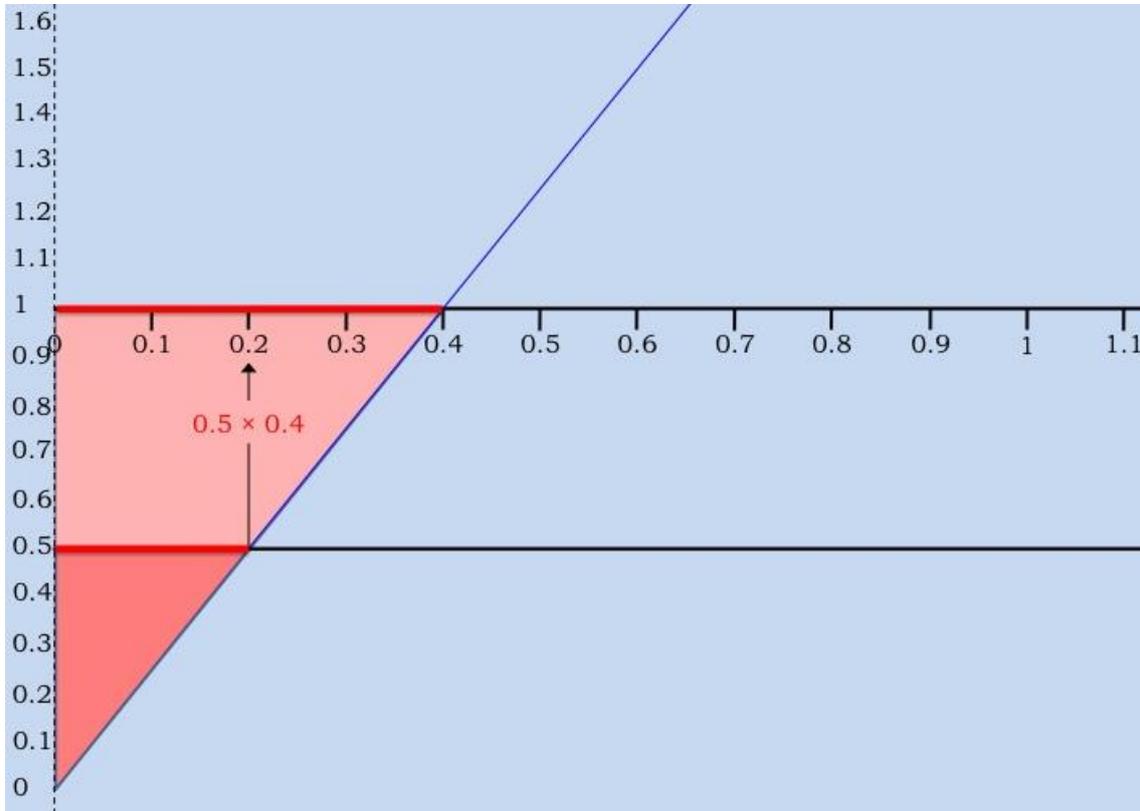
Some pupils may like to extend their images into negative numbers in both directions ...



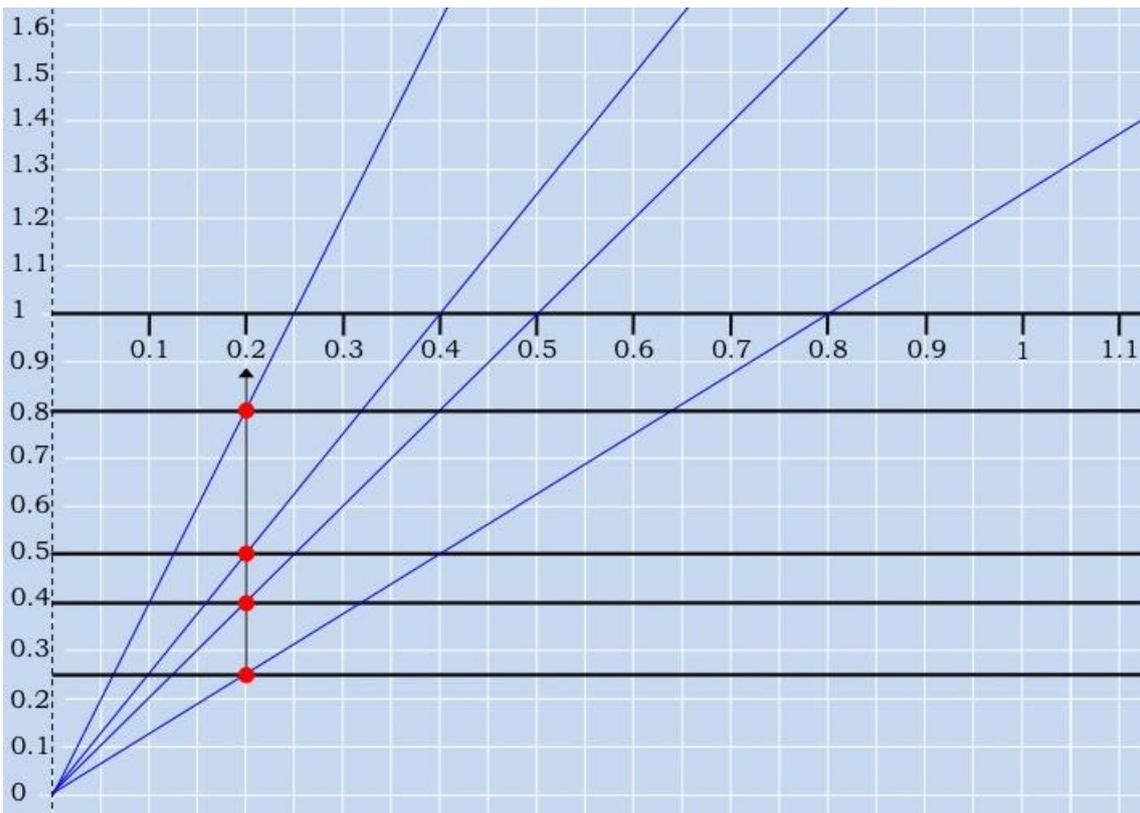
...and confirm relationships between products involving negative numbers, which some pupils might be able to show generally:



A strength of these kinds of image is that pupils can use them to focus on multiplication of numbers between 0 and 1. This image should remind pupils of the starting point for these scaling-images ...



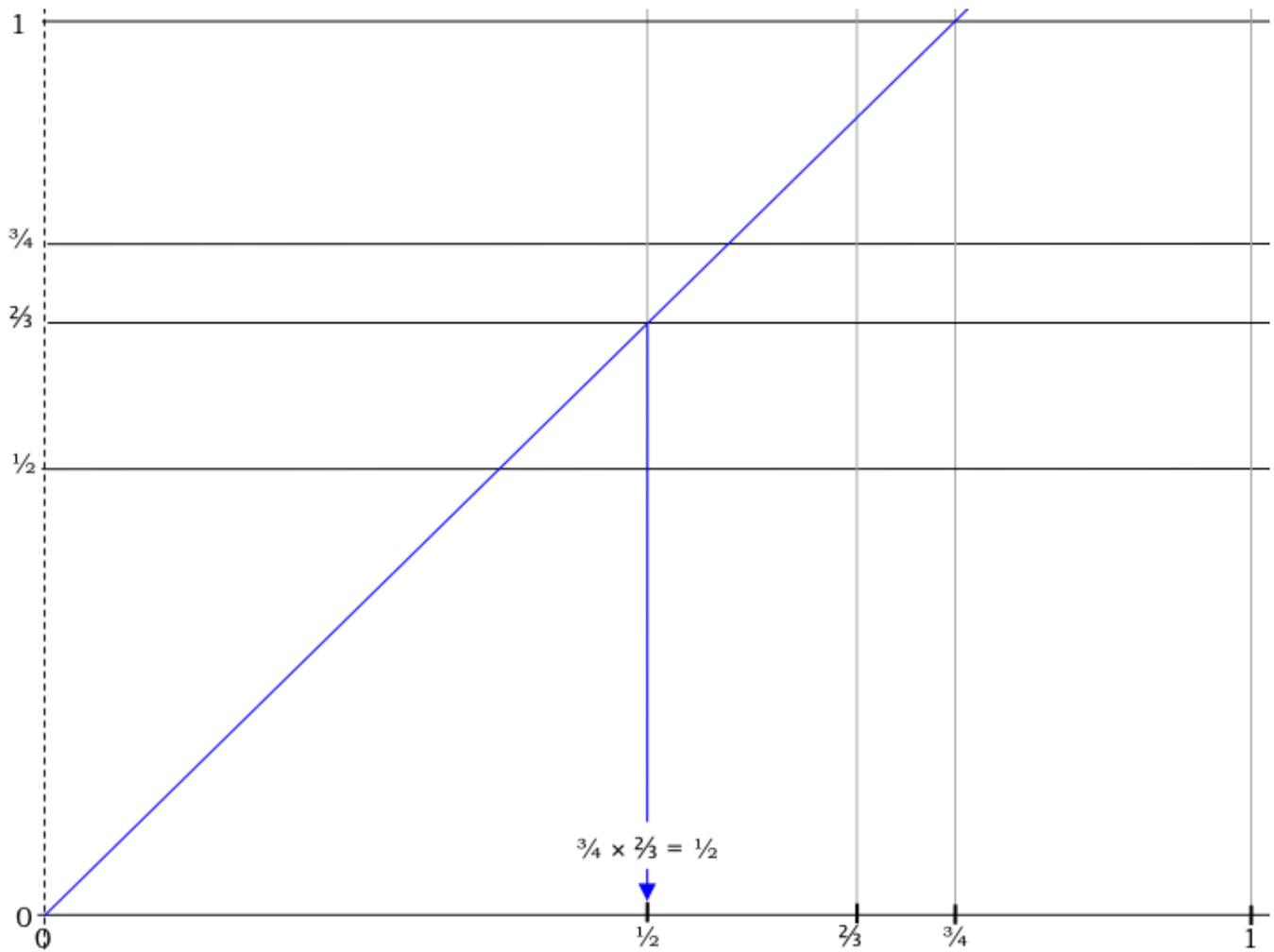
... and this example ...



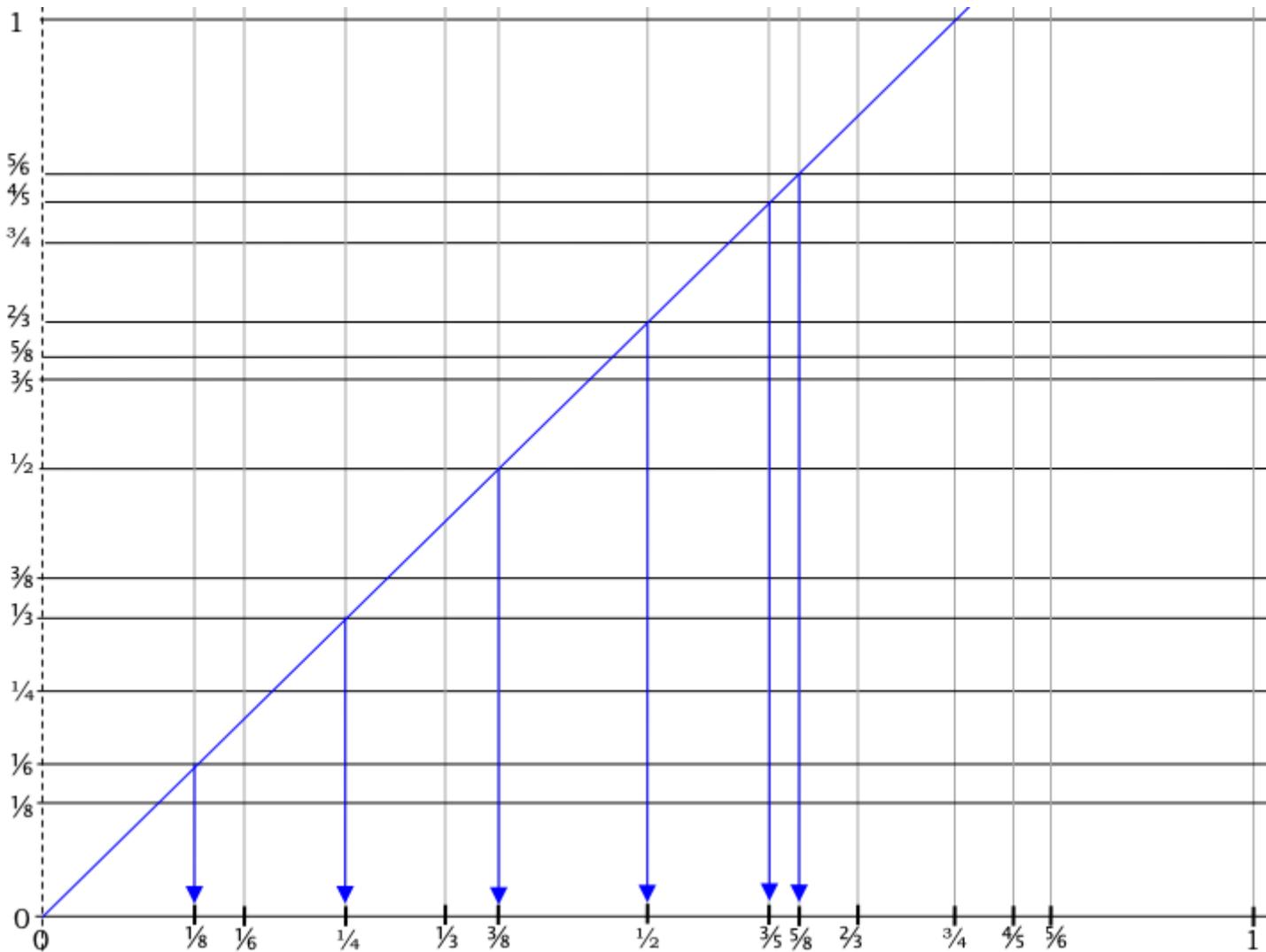
... shows that  $0.8 \times 0.25 = 0.5 \times 0.4 = 0.4 \times 0.5 = 0.25 \times 0.8 = 0.2$ .

Could your pupils draw a similar diagram that shows some products (of two numbers) equal to 0.4 or 0.6?

Scaling-diagrams that show multiplication of fractions can be very revealing to pupils. This is a typical example ...



... and this next image shows the result of multiplying various fractions by  $\frac{3}{4}$  ...



Could your pupils draw a similar diagram to show multiplication of some fractions by, say,  $\frac{2}{3}$  or  $\frac{5}{8}$ ?

You can find previous *It Stands to Reason* features [here](#)

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## Eyes Down

### Have I assumed too much?

I've been stumped as to why so many students in my year 10 group failed to answer this question correctly:

(a) Use your calculator to work out  $\frac{38.5 \times 14.2}{18.4 - 5.9}$ .

Write down all the figures on your calculator display.  
You must give your answer as a decimal.

43.736

(b) Write your answer to part (a) correct to 1 significant figure.

43.7

I know to expect students to confuse rounding to a given number of significant figures with rounding using decimal places and until now they were always areas that I thought that I was pretty good at teaching – students were able to quickly recognise why they weren't awarded accuracy marks in an assessment and rarely made the same mistake twice as I'd make a point of highlighting the misconception and it would result in a real "facepalm" moment for the student.

However, I only started teaching the vast majority of this group at the start of the academic year so they have different starting points and I think I made the mistake of over-estimating their ability with what I consider to be basic "maths". I had assumed that they would be fluent in the two different rounding strategies and because of this I haven't been regularly reinforcing the "rules" and "special cases" through rounding answers to varying degrees of accuracy when working on other areas of Maths as I would have done had I taught the topic from first principles.

Luckily I'm going to be able to rectify this quickly as we move to focusing on "error intervals" I will be able to spend a few lessons looking at "where", "when" and "why" the two different methods of rounding are used. It is important to remember that in some cases, the students will remember "rounding rules" from primary school which may need unravelling – it isn't uncommon for students to think that it is always appropriate to round to the same number of decimal places (usually 2) if the level of accuracy used is always the same. In terms of significant figures I like to use number lines or scales in the first instance as I've found that it is easy to confuse students if all the rules are presented to them at the same time, especially given that there are special cases to consider.

Wish me luck!



*If you have a thought-inducing picture, please send a copy (ideally, about 1-2Mb) to us at [info@ncetm.org.uk](mailto:info@ncetm.org.uk) with 'Secondary Magazine Eyes Down' in the email subject line. Include a note of where and when it was taken, and any comments on it you may have. If your picture is published, we'll send you a £20 voucher.*

You can find previous *Eyes Down* features [here](#)

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