MathsHUBS
Resources

National Centre
for Excellence in the
Teaching of Mathematics

Mastery Professional Development

1 The structure of the number system

PDF

1.2 Properties of number

Guidance document | Key Stage 3

Making connections

The NCETM has identified a set of six 'mathematical themes' within Key Stage 3 mathematics that bring together a group of 'core concepts'.

The first of these themes is *The structure of the number system*, which covers the following interconnected core concepts:

1.1   Place value, estimation and rounding

1.2   **Properties of number**

1.3   Ordering and comparing

1.4   Simplifying and manipulating expressions, equations and formulae

This guidance document breaks down core concept *1.2 Properties of number* into three statements of knowledge, skills and understanding:

1.2.1   Understand multiples

1.2.2   Understand integer exponents and roots

1.2.3   Understand and use the unique prime factorisation of a number

Then, for each of these statements of knowledge, skills and understanding we offer a set of key ideas to help guide teacher planning.

Please note that these materials are principally for professional development purposes. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

Overview

Students will have been introduced to multiples and factors at Key Stage 2 and will have had the opportunity to find factor pairs for a given number. They should know that prime numbers have exactly two factors and why, therefore, one is not prime. They should also be able to recall prime numbers up to 19 and identify others (possibly using the Sieve of Eratosthenes to find all the prime numbers up to 100).

Students will have found common factors and multiples for pairs of numbers, and it is likely that they will have done this by making lists of factors and multiples and looking for common items. The shift at Key Stage 3 is to examine the structure of the numbers involved and explore ways of representing them, for example, by using factor trees and Venn diagrams. In particular, expressing numbers as the product of prime factors will enable students to reason about and identify highest common factors and lowest common multiples, and to appreciate this as a more efficient method than listing in some situations.

Students should already be able to recognise square and cube numbers, and use appropriate notation, from their work at Key Stage 2. At Key Stage 3, they will build on this by using other positive integer exponents greater than three, and associated real roots (square, cube and higher). Work on exponents and roots in Key Stage 3 provides the foundation for future learning when exploring negative integer and fractional exponents in Key Stage 4.

Prior learning

Before beginning to teach *Properties of number* at Key Stage 3, students should already have a secure understanding of the following from previous study:

Key stage	Learning outcome
Upper Key Stage 2	• Identify multiples and factors, including finding all factor pairs of a number, and common factors of two numbers • Know and use the vocabulary of prime numbers, prime factors and composite numbers (non-prime, greater than one) • Establish whether a number up to 100 is prime and recall prime numbers up to 19 • Recognise and use square numbers and cube numbers, and the notation for squared (^{2}) and cubed (^{3}) • Solve problems involving multiplication and division, including using their knowledge of factors and multiples, squares and cubes • Identify common factors, common multiples and prime numbers • Use common factors to simplify fractions; use common multiples to express fractions in the same denomination

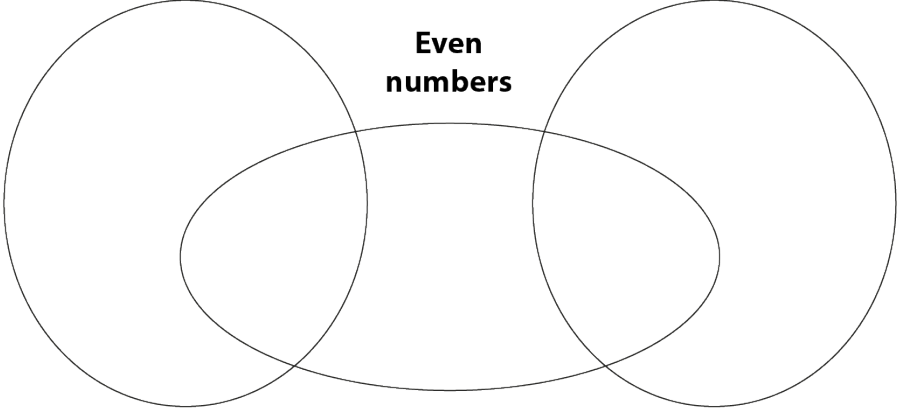You may find it useful to speak to your partner schools to see how the above has been covered and the language used.

You can find further details regarding prior learning in the following segment of the NCETM primary mastery professional development materials[1]:

- Year 5: 2.21 Factors, multiples, prime numbers and composite numbers

Checking prior learning

The following activities from the NCETM primary assessment materials[2] and the Standards & Testing Agency's past mathematics papers[3] offer useful ideas for assessment, which you can use in your classes to check whether prior learning is secure:

Reference	Activity
Year 5 page 15	8 is a multiple of 4 and a factor of 16 6 is a multiple of 3 and a factor of ☐ ☐ is a multiple of 5 and a factor of ☐ ☐ is a multiple of ☐ and a factor of ☐
Year 5 page 16	Fill in the missing numbers in this multiplication pyramid. 108 / 6 / 3 / 2
Year 6 page 19	In each number sentence, replace the boxes with different whole numbers less than 20 so that the number sentence is true: $\dfrac{1}{\square} = \dfrac{3}{\square}$ $\quad$ $\square \div \square = \square \,.\, \square$ $\dfrac{\square}{3} = \dfrac{\square}{12}$ $\quad$ $\dfrac{30}{\square} = \dfrac{45}{\square}$ $\dfrac{\square}{\square} = \dfrac{\square}{\square}$

2016 Key Stage 2 Mathematics Paper 1: arithmetic Question 19	$3^2 + 10 = \boxed{}$
2016 Key Stage 2 Mathematics Paper 2: reasoning Question 5	Write each number in its correct place on the diagram. 16 17 18 19 **Prime numbers** **Square numbers** **Even numbers**
2016 Key Stage 2 Mathematics Paper 2: reasoning Question 14	Write **all** the common multiples of 3 and 8 that are **less than 50**.
2017 Key Stage 2 Mathematics Paper 3: reasoning Question 8	Write three factors of 30 that are **not** factors of 15. $\boxed{}$ $\boxed{}$ $\boxed{}$
2017 Key Stage 2 Mathematics Paper 3: reasoning Question 18	A **square** number and a **prime** number have a total of 22. What are the two numbers? $\boxed{}$ + $\boxed{}$ = 22 square number prime number

2018 Key Stage 2 Mathematics Paper 1: arithmetic Question 32	$9^2 - 36 \div 9 = \boxed{}$
2018 Key Stage 2 Mathematics Paper 3: reasoning Question 5	Tick the numbers that are common factors of both **12 and 18**. 2 $\boxed{}$ 3 $\boxed{}$ 6 $\boxed{}$ 9 $\boxed{}$ 12 $\boxed{}$

Key vocabulary

Term	Definition
cube root	A value or quantity whose cube is equal to a given quantity. Example: the cube root of 8 is 2 since $2^3 = 8$. This is recorded as $\sqrt[3]{8} = 2$ or $8^{\frac{1}{3}} = 2$.
exponent	Also known as 'index', a number, positioned above and to the right of another (the base), indicating repeated multiplication when the exponent is a positive integer. Example 1: n^2 indicates $n \times n$; and 'n to the (power of) 4', that is n^4 means $n \times n \times n \times n$. Example 2: since $2^5 = 32$ we can also think of this as '32 is the fifth power of 2'. Any positive number to the power of 1 is the number itself; $x^1 = x$, for any positive value of x. Exponents may be negative, zero, or fractional. Negative integer exponents are the reciprocal of the corresponding positive integer exponent, for example, $2^{-1} = \frac{1}{2}$. Any positive number to the power of zero equals 1; $x^0 = 1$, for any positive value of x.

	The positive unit fractional powers represent roots, which are the inverse to the corresponding integer powers; thus $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$, since $2^3 = 8$.
	Note: power notation is not used for zero, since division by zero is undefined.
highest common factor (HCF)	The common factor of two or more numbers which has the highest value.
	Example: 16 has factors 1, 2, 4, 8, 16.
	24 has factors 1, 2, 3, 4, 6, 8, 12, 24.
	56 has factors 1, 2, 4, 7, 8, 14, 28, 56.
	The common factors of 16, 24 and 56 are 1, 2, 4 and 8.
	Their highest common factor is 8.
index	Also known as 'exponent', a number, positioned above and to the right of another (the base), indicating repeated multiplication when the index is a positive integer.
lowest common multiple (LCM)	The common multiple of two or more numbers, which has the lowest value.
	Example: 3 has multiples 3, 6, 9, 12, 15, 18, 21, 24 …
	4 has multiples 4, 8, 12, 16, 20, 24 …
	6 has multiples 6, 12, 18, 24, 30 …
	The common multiples of 3, 4 and 6 include 12, 24 and 36.
	The lowest common multiple of 3, 4 and 6 is 12.
	Also known as the 'least common multiple'.
prime factor decomposition	The process of expressing a number as the product of factors that are prime numbers.
	Example: $24 = 2 \times 2 \times 2 \times 3$ or $2^3 \times 3$.
	Every positive integer has a unique set of prime factors.
square root	A number whose square is equal to a given number.
	Example: one square root of 25 is 5 since $5^2 = 25$. The square root of 25 is recorded as $\sqrt{25} = 5$. However, as well as a positive square root, 25 has a negative square root, since $(-5)^2 = 25$.
Venn diagram	A simple visual diagram used to describe the relationships between two sets.
	With two or three sets, each set is often represented by a circular region. The intersection of two sets is represented by the overlap region between the two sets.
	With more than three sets, Venn diagrams can become very complicated.
	A rectangle is usually drawn around the diagram to represent the universal set, i.e. the overall set of which the 2 (or 3) sets are a subset.

Collaborative planning

Below we break down each of the three statements within *Properties of number* into a set of key ideas to support more detailed discussion and planning within your department. You may choose to break them down differently depending on the needs of your students and timetabling; however, we hope that our

suggestions help you and your colleagues to focus your teaching on the key points and avoid conflating too many ideas.

Please note: We make no suggestion that each key idea represents a lesson. Rather, the 'fine-grained' distinctions we offer are intended to help you think about the learning journey irrespective of the number of lessons taught. Not all key ideas are equal in length and the amount of classroom time required for them to be mastered will vary, but each is a noteworthy contribution to the statement of knowledge, skills and understanding with which it is associated.

The following letters draw attention to particular features:

D Suggested opportunities for **deepening** students' understanding through encouraging mathematical thinking.

L Examples of shared use of **language** that can help students to understand the structure of the mathematics. For example, sentences that all students might say together and be encouraged to use individually in their talk and their thinking to support their understanding (for example, *'The smaller the denominator, the bigger the fraction.'*).

R Suggestions for use of **representations** that support students in developing conceptual understanding as well as procedural fluency.

V Examples of the use of **variation** to draw students' attention to the important points and help them to see the mathematical structures and relationships.

PD Suggestions of questions and prompts that you can use to support a **professional development** session.

For selected key ideas, marked with an asterisk (*), we exemplify the common difficulties and misconceptions that students might have and include elements of what teaching for mastery may look like. We provide examples of possible student tasks and teaching approaches, together with suggestions and prompts to support professional development and collaborative planning. You can find these at the end of the set of key ideas.

Key ideas

1.2.1 Understand multiples

Students should be familiar with the term 'multiple' from their work in Key Stage 2. They should be able to recognise whether a number is a multiple of another positive integer by recalling the lists of multiples or counting on multiples from the relevant times table.

The focus at Key Stage 3 is on examining the structure of numbers and being able to reason whether numbers are multiples of other numbers or not without the need for creating lists of multiples. For example, students should recognise that 176 is a multiple of eight because it is the sum of 160 and 16, both of which are multiples of eight. Connections can be made here to the rules for divisibility, with students exploring why the rules work and how they can help identify multiples of a number.

1.2.1.1   Understand what a multiple is and be able to list multiples of *n*

1.2.1.2* Identify and explain whether a number is or is not a multiple of a given integer

1.2.2 Understand integer exponents and roots

Students should already be familiar with at least the first 12 square numbers and may be familiar with a range of cube numbers (1^3 to 5^3) from their work at Key Stage 2. They are likely to have a

basic grasp of the notation, including square and cube roots, and know that, for example, $\sqrt{16} = 4$ because $4^2 = 16$ and $\sqrt[3]{8} = 2$ because $2^3 = 8$. Students should recognise that the square (or cube) root of any number can be found, but that it is only when they are perfect square (or cube) numbers that this operation will give an integer solution.

In Key Stage 3, students will need to explore positive integer exponents greater than three. This will support other Key Stage 3 work involving writing numbers as the product of prime factors in simplified terms, thus enabling identification of the highest common factor and the lowest common multiple of two or more positive integers.

1.2.2.1   Understand the concept of square and cube

1.2.2.2   Understand the concept of square root and cube root

1.2.2.3   Understand and use correct notation for positive integer exponents

1.2.2.4   Understand how to use the keys for squares and other powers and square root on a calculator

1.2.3   Understand and use the unique prime factorisation of a number

Finding factors of a number will be familiar from Key Stage 2. Students should be able to find factor pairs for a given number and know that a number which has exactly two factors is prime. Students are expected to recall prime numbers up to 19 and be able to establish prime numbers up to 100. The focus in this set of key ideas is to be able to identify factors and prime numbers based on a deep understanding of number structure. Where rules for divisibility are used to help these processes, the focus should be on understanding why these rules work.

Students' experience of highest common factors and multiples at Key Stage 2 is likely to be limited to their work on simplifying fractions and checking to see if they have found the greatest number that is a factor of both the numerator and denominator. Similarly, when expressing fractions in the same denomination in order to compare them, for example, students may have identified the least common multiple of the two denominators even if this formal term has not been used.

In Key Stage 3, students will come across the unique prime factorisation property for the first time. Students will need to recognise that any positive integer greater than one is either a prime number itself or can be expressed as a product of prime numbers, and that there is *only one* way of writing a number in this way. It is this property that will help students to identify efficiently the highest common factor and lowest common multiple for two or more positive integers.

1.2.3.1   Understand what a factor is and be able to identify factors of positive integers

1.2.3.2   Understand what a prime number is and be able to identify prime numbers

1.2.3.3   Understand that a positive integer can be written uniquely as a product of its prime factors

1.2.3.4*   Use the prime factorisation of two or more positive integers to efficiently identify the highest common factor

1.2.3.5   Use the prime factorisation of two or more positive integers to efficiently find their lowest common multiple

Exemplified key ideas

1.2.1.2 Identify and explain whether a number is or is not a multiple of a given integer

Common difficulties and misconceptions

Students often find multiples of an integer by listing numbers in the specified times table. This strategy is efficient for small numbers of multiples but can lead to misconceptions, such as thinking that numbers have only 12 multiples or that numbers outside of the times tables do not have multiples.

Students need to be able to identify the patterns present in multiples of an integer and explore the structures which generate those patterns. For example, students should understand that adding two different multiples of the same number results in another multiple of that number. Similarly, that if a number is a multiple of 15, for example, it is also a multiple of five and of three. By exploring multiples and reasoning in this way, students can decide whether any number is or is not a multiple of a given integer.

Strategies for identifying multiples usually link to division, especially for larger numbers which are not multiples known from multiplication tables. Students who find it challenging to make the connection between the idea of multiples (numbers in multiplication tables) and division may benefit from revisiting prior work on $factor_a \times factor_b = product$, and variations of this: $product \div factor_a = factor_b$

The use of partitioning can also be a useful strategy when identifying multiples. As shown in *Example 4,* below, 6 132 is not a multiple of eight because $6\,132 = 6\,000 + 120 + 12$ and while 6 000 and 120 are both multiples of eight, 12 is not. Using divisibility rules to test whether a number is a multiple or not may also be helpful, but if using these, students should be given time to investigate both why they work and how they can be used.

Students should also understand the connections between multiplication tables. For example, students should know that all multiples of ten are multiples of five but not all multiples of five are multiples of ten. The use of a multiplication grid may support students to see these connections, consider the structures behind them and, consequently, be able to reason fully.

If students have only experienced multiples as a list of positive integers, defining multiples by a generalised statement, such as, *'For any integers a and b, a is a multiple of b if a third integer c exists so that a = bc'* will help students understand that 14, 49, 70 and −21 are all multiples of seven because $14 = 7 \times 2$, $49 = 7 \times 7$, $70 = 7 \times 10$ and $-21 = 7 \times -3$.

What students need to understand	Guidance, discussion points and prompts
Identify numbers which are and are not multiples of 2, 5 or 10. *Example 1:* a) *Place a tick (✓) in the cell if the number is a multiple of 2, 5 or 10. Explain how you know.*	**V** *Example 1* has been designed to draw students' attention to patterns in multiplication tables. Students should be familiar with their two, five and ten multiplication tables, so this example gives an opportunity to explore the relationships between the multiples and the structures behind them.

a) *Place a tick (✓) in the cell if the number is a multiple of 2, 5 or 10. Explain how you know.*

	2	5	10
830			
457			
12 974			
60 535			
519 276			

b) *What general statements can you make about multiples of 2, 5 and 10?*

c) *Find an integer which is a common multiple of 2, 5 and 10. Can you find another? And another? What do you notice?*

V *Example 1* has been designed to draw students' attention to patterns in multiplication tables. Students should be familiar with their two, five and ten multiplication tables, so this example gives an opportunity to explore the relationships between the multiples and the structures behind them.

Asking students to explain their reasoning in many different ways (for example, an integer is a multiple of two if: '*It is an even number*', '*It has a 1s digit which is 0, 2, 4, 6, 8*', '*When halved, the quotient is an integer*', etc.) can help students to generalise the structure of such numbers.

L There is an opportunity here to encourage the use of generalised statements with precise language, such as:

'*An integer is a multiple of two if …*'

'*An integer is a multiple of five if …*'

'*An integer is a multiple of ten if …*'

You may wish to support students in generalising this idea by offering the following sentence structure:

'*a is a multiple of b if a third integer c exists so that a = bc.*'

PD How do you define a multiple? Do some of your students think multiples are the products listed in a times table? Asking students whether 91 (the 13th multiple of seven and a multiple that fails all the 'common' divisibility tests) is a prime number or not is a useful activity for uncovering any possible misconceptions your students might have.

Example 2:

a) *The following four-digit number is a multiple of 2 but not a multiple of 5.*

  4   *1*   *7*   ☐

  What could the 1s digit be? How many possibilities are there?

b) *The following four-digit number is a multiple of 5 but not a multiple of 2.*

  4   *1*   *7*   ☐

  What could the 1s digit be? How many possibilities are there?

c) *What is the biggest four-digit number which is both a multiple of 2 and a multiple of 5?*

L   *Example 2* requires students to apply their understanding to solve problems involving multiples. Asking students to find all the possibilities will help them to generalise.

PD Can you construct other 'all possibilities' examples that might highlight the importance of the rules for divisibility?

Example 3:

a) *Guy says that 543 210 is a multiple of 5 because it is a multiple of 10. Is Guy correct? Justify your answer.*

b) *Harriet says that 12 345 is a multiple of 10 because it is a multiple of 5. Is Harriet correct? Justify your answer.*

c) *Is there a similar relationship between other pairs of multiples, for example, 3 and 9? Can you explain why this is the case?*

V   The numbers in *Example 3* are large in order to encourage students to draw conclusions by thinking about the structure of the numbers rather than performing calculations. The questions are very similar in style, so students will need to carefully notice the difference and consider what is being asked of them.

R   The use of a multiplication grid may support students to identify patterns and consider the structure behind them. It may also be useful in identifying other integers with similar relationships.

D   This example provides an opportunity for students to generalise using algebra, e.g. $10x = 5(2x)$, etc. This may also support them in identifying the structures present.

PD Can you construct other 'what it's not' examples that might highlight the importance of the structure of the numbers?

Identify numbers which are and are not multiples of 2, 4 or 8.

Example 4:

a) *Place a tick (✓) in the cell if the number is a multiple of 2, 4 or 8. Explain how you know.*

	2	**4**	**8**
44			
96			
125			
6 132			
319 456			

b) *How can partitioning the number into known multiples of 4 and 8 help determine whether the number is a multiple of 4 or 8?*

c) *Raj says that 264 is **not** a multiple of 8 because 264 = 100 + 100 + 64 and although 64 is a multiple of 8, 100 is not. Explain why Raj is wrong.*

d) *Can you extend this idea to find a divisibility test for 4 and 8? Explain to your partner why it works.*

V In *Example 4*, the numbers get progressively bigger so that students can start with something they feel comfortable with (most will recognise 44 as a multiple of two and four but not eight) but will quickly need to generalise in order to check without calculating. Part c) is important as students need to appreciate that, when using partitioning, there should only be one element which is not a multiple of the specified number.

PD It is worth considering how you would work through this example with a class to ensure students 'go deeper' with their thinking. What prompts could you give them? Can you think of an example which might help?

D *Example 4* is a great opportunity for students to think deeply about the divisibility rules and why they work. Connections to prior learning of partitioning might help students to reason. For example, with 6 132, students may realise that 6 000 is a multiple of eight (because 1 000 is), 32 is a multiple of eight, but 100 is not. A deeper level of thinking is to use partitioning to generalise the various 'rules of divisibility'. For example, 100 is a multiple of four, so multiples of 100 will always be multiples of four. Therefore, when considering whether an integer is a multiple of four, only the 10s and 1s digits need to be considered. This reasoning can be extended to the divisibility test for multiples of eight.

Identify numbers which are and are not multiples of 3, 6 or 9.

Example 5:

Place a tick (✓) in the cell if the number is a multiple of 3, 6 or 9. Explain how you know.

	3	6	9
42			
99			
186			
40 311			
162 504			

R In *Example 5*, students may benefit from exploring patterns in multiples of nine. One representation which might support their thinking is a multiplication grid. Recognising that the digits in multiples of nine always sum to nine (digital root of nine) is fundamental to the divisibility test for nine.

By partitioning (for example) a three-digit number *abc* and writing it as $100a + 10b + c$, students can see that this can be rewritten as $99a + 9b + (a + b + c)$. Since $99a$ and $9b$ are always multiples of nine, the only check required is whether the digital root ($a + b + c$) is divisible by nine. Students could then extend this to explore and reason about the divisibility test for three.

V *Example 5* is designed to draw students' attention to the relationships between the multiples of three, six and nine. Students should notice that multiples of nine will always be multiples of three, but multiples of three are not always multiples of nine. They should also notice that some multiples of three are also multiples of six, but only the even ones. This connects to learning in the next example (*Example 6*).

Make connections between multiples of integers that are 10 or less.

Example 6:

*If a number is a multiple of the integer indicated *, what else must it also **always** be a multiple of?*

Complete the table below by indicating (with a tick) other numbers it is a multiple of.

	2	3	4	5	6	7	8	9	10
a)									*
b)					*				
c)						*			
d)								*	
e)			*						
f)							*		

V In *Example 6*, students must consider the structures and apply their knowledge of multiplicative relationships.

- Part a): If an integer is a multiple of ten, it is always a multiple of both two and five. This first example is one that students have explored previously.
- Part b): If an integer is a multiple of six, it is always a multiple of two and three. It is worth drawing students' attention to the converse of this: to check if an integer is a multiple of six, it is often easier to check it is a multiple of two and three.
- Part c): If an integer is a multiple of seven, it is not always a multiple of any other integer (except one). Why is one not included in this table?
- Part d): If an integer is a multiple of nine, it is always a multiple of three, but not necessarily a multiple of six. In *Example 5*, students examined the connections between divisibility tests for nine and three.
- Part e): If an integer is a multiple of four, it is always a multiple of two, but not necessarily a multiple of eight; it's worth drawing attention to this.
- Part f): If an integer is a multiple of eight, then it will always be a multiple of both two and four. In part b) students will have noticed that if an integer is a multiple of both two and three, then it will be a multiple of six. Why, then, is it not true to say that if an integer is a multiple of both two and four, then it will be a multiple of eight?

D Students could be challenged by asking further questions, such as, 'If a number is a product of a multiple of five and a multiple of eight, what else must it be a multiple of? How do you know?'

Solve problems where there is more than one answer and there are elements of experimentation, investigation, checking, reasoning, proof, etc.

Example 7:

a) *Answer these statements using: Always, Sometimes or Never.*

   (i)   *If a number is a multiple of 10, it is also a multiple of 5.*

   (ii) *If a number is a multiple of 4, it is also a multiple of 8.*

   (iii) *If a number is a multiple of 9, it is also a multiple of 2.*

   (iv) *Multiples are positive integers.*

b) *Is it always, sometimes or never true that adding two consecutive multiples of 5 will give a multiple of 10?*

c) *Is it always, sometimes or never true that adding five consecutive multiples of 2 will give a multiple of 10?*

For students who have demonstrated a secure understanding of identifying multiples, you could encourage them to go deeper by solving more complex problems, such as exploring all possibilities, creating their own examples and testing conjectures.

V It is important that students are given opportunities to investigate multiplicative relationships. *Example 7* provides a structure for this.

Students need to reason fully and should provide examples and counter-examples to demonstrate their mathematical understanding.

PD How can you build a classroom culture where students feel comfortable and confident to express their opinions, debate and challenge each other?

How do you teach learning behaviours so that students justify their answers as a matter of habit rather than requiring prompting?

Solve familiar and unfamiliar problems, including real-life applications.

Example 8:

Two lighthouses flash at different intervals. One flashes every 5 seconds and the other every 8 seconds.

At exactly midnight (00:00:00) they flash together. When will they next flash at the same time?

D Problems such as *Example 8* provide opportunities for students to practise their understanding of a concept (i.e. intelligent practice rather than mechanical repetition) and to focus on relationships, not just the procedure.

PD Can you create some other problems and contexts suitable for your classes where the need to find multiples is relevant?

1.2.3.4 Use the prime factorisation of two or more positive integers to efficiently identify the highest common factor

Common difficulties and misconceptions

Fundamental to this concept is that students are at ease with the idea that $2 \times 3 \times 5$ is just another way of expressing the number 30 and does not need to be calculated. In fact, by leaving it in this form, it is much easier to discern factors and (when there are two numbers expressed in this way) to discern their common factors. However, students often struggle with this idea. When asked whether $2 \times 3 \times 5$ is a multiple of ten or not, it is not uncommon for students to multiply the three factors together to obtain 30 before they are able to say that it is a multiple of ten.

An important awareness in this key idea is that when two numbers are written as the product of two or more factors, looking for overlaps between the two products helps to find a common factor.

For example, by writing $30 = 5 \times 6$ and $105 = 5 \times 21$, it is easily seen that five is a common factor.

However, we cannot be sure that five is the *highest* common factor unless each number is written as the product of prime factors, e.g. $30 = 2 \times 3 \times 5$ and $105 = 3 \times 5 \times 7$. When written in this format, it is clear that $3 \times 5 = 15$ is the highest common factor.

Students may experience difficulties when the product of repeated prime factors is expressed using index notation (e.g. $450 = 2 \times 3^2 \times 5^2$ and $1\,500 = 2^2 \times 3 \times 5^3$), as it may be harder to detect the common factor of $2 \times 3 \times 5^2$ in this form. It will be important for students to have experience of discerning highest common factors from both the index and non-index form to help avoid this difficulty.

Students will have already experienced Venn diagrams in Key Stage 2; however, using them to record prime factors, and for revealing that the highest common factor is the product of the prime factors in the intersection, is likely to be an unfamiliar idea. The example above could be displayed in the Venn diagram below, showing that the highest common factor of 450 and 1 500 is $2 \times 3 \times 5^2$, or 150.

Prime factors of 450   **Prime factors of 1 500**

$450 = 2 \times 3^2 \times 5^2$ $1\,500 = 2^2 \times 3 \times 5^3$
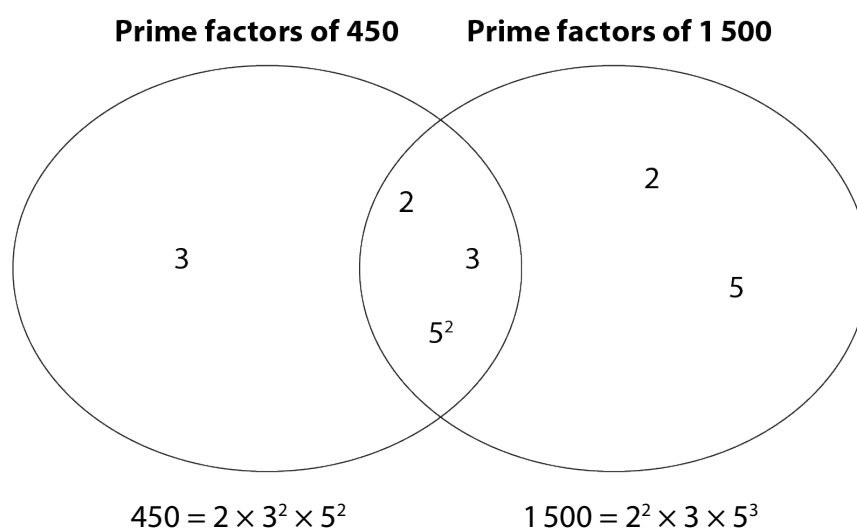
Exploring multiple methods (such as listing factors, using the prime factorisation, using a Venn diagram, etc.) and establishing which is most efficient for numbers of varying sizes is important in this key idea. Discussing and comparing different approaches and solutions will support students in identifying and choosing appropriate and efficient methods.

Encouraging students to apply this idea to algebraic expressions, for example:

What is the highest common factor of p^2qr^3 and pq^2r^2?

will support them in generalising this idea and developing a deep and secure understanding of the underpinning mathematical structure.

What students need to understand	Guidance, discussion points and prompts

Find the highest common factor of two numbers when they have been written as the product of their prime factors.
 	
Example 1:
 	
Find the highest common factor of these pairs of numbers.
 	
a) $10 = 2 \times 5$
     $6 = 2 \times 3$
 	
b) $12 = 2 \times 2 \times 3$
     $20 = 2 \times 2 \times 5$
 	
c) $30 = 2 \times 3 \times 5$
     $70 = 2 \times 5 \times 7$
 	
d) $60 = 2 \times 2 \times 3 \times 5$
     $90 = 2 \times 3 \times 3 \times 5$
 	
e) $42 = 2 \times 3 \times 7$
     $210 = 2 \times 3 \times 5 \times 7$
 	
f) $29 = 29$
     $16 = 2 \times 2 \times 2 \times 2$
	V Choosing small numbers allows students to find the highest common factor using multiple methods (both by listing factors of each number and by considering prime factors). In *Example 1*, the numbers have already been written as a product of their prime factors so that students can focus on finding the highest common factor and not on prime factorisation, although they should realise that this is a previous step in this method.
	
	Part of the purpose of an exercise like this is for students to explore the most efficient method and understand why, when numbers are written as product of prime factors, the combination of their common prime factors will generate the highest common factor.
	
	• Part a) has been written so that the numbers only share one common prime factor (two).
	• Part b) is designed to ensure that students can identify and understand that the numbers share two common prime factors. Comparing the repeated common prime factor of two, to the common factors from the lists (two and four), is also worthwhile so that students understand that it is the combination of the prime factors that is important.
	• In part c), the two common prime factors are different and lead to a highest common factor of ten. Again, comparison with common factors of two, five and ten from their list of factors is important.
	• The numbers in part d) have been chosen as they share three common prime factors.
	• In part e), the highest common factor is one of the numbers in its entirety.
	• Part f) has been chosen to facilitate discussion about what to do if the highest common factor is one, as this is not always

apparent when numbers are written as the product of their prime factors.

PD Asking students to identify all common factors and then linking this to all common prime factors, before progressing to find the highest common factor, may be helpful in supporting their understanding. How could you connect the learning in this key idea to previous work on factors?

Exploring the efficiency of different methods is a significant element of developing fluency. Encouraging students to share their methods and discuss the advantages and disadvantages of each, as well as when to select an appropriate method, will help prevent the blind application of a procedure without conceptual consideration. To what extent do you already do this in your classrooms? How and when is best to facilitate classroom discussion?

Example 2:

a) *Find the highest common factor of this pair of numbers, where p, q, r, s and t are prime:*

$x = p \times q \times r$

$y = p \times r \times s \times t$

b) *Why do p, q, r, s and t have to be prime for this to work?*

D Asking students to apply a procedure to algebraic examples is a good way to assess the depth of their understanding. Encouraging students to generalise a concept is beneficial in supporting them in dealing with problems that are based on the same concept but presented differently, for example, finding the highest common factor of three numbers.

Challenging students to write an algebraic example which mirrors *Example 1* part f) may also be a useful addition here.

Find the highest common factor of two numbers when they have been written as the product of their prime factors and simplified using indices.

Example 3:

a) *Gosia thinks the highest common factor of 72 and 180 is 6. Her working is below:*

$72 = 2^3 \times 3^2$

$180 = 2^2 \times 3^2 \times 5$

$HCF = 2 \times 3 = 6$

Do you agree with Gosia? Justify your answer.

PD Students need to be fluent at simplifying expressions using indices. How can you assess and support students in their understanding of indices? What connections can you make to prior learning?

V *Example 3* gives students an opportunity to explore misconceptions surrounding the use of indices and, consequently, understand the need to take indices into account when finding the highest common factor.

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b) $x = a^2 \times b \times c^2$

$y = a \times b^3 \times c^2$

where a, b and c are prime.

Harrison thinks the highest common factor of x and y is $a \times b \times c^2$.

Do you agree with Harrison? Justify your answer.

In part a), the numbers are large enough to encourage students to move away from listing factors and instead compare prime factors to find the highest common factor. Asking students to find all common prime factors by deconstructing the simplified products may support them in finding the highest common factor.

In part b), the concept has been applied to algebraic prime factors. Having discussed part a), students then have the opportunity to demonstrate understanding in a more generalised form.

D Challenge students to think deeply about the concept by asking them to create questions of their own that meet specific criteria. For example:

- Write down two numbers which have a highest common factor of …
- Write down two numbers which have a highest common factor of one.
- $72 = 2^3 \times 3^2$
  $180 = 2^2 \times 3^2 \times 5$
  What is the highest common factor of 72 and 180?
  Find an additional number so that the highest common factor of the three numbers (72, 180 and the new number) remains the same.
  Write down the prime factorisation of a third number so that all three numbers have a highest common factor that is less than 12.
  Write down a number with a highest common factor of 12 when paired with 72, but a highest common factor that is greater than 12 when paired with 180.

PD Can you construct other 'what it's not' examples that might highlight the importance of the use of indices?

Find the highest common factor of a set of numbers when they have been written as the product of their prime factors in simplified form.

Example 4:

Find the highest common factor of these sets of numbers.

a) $20 = 2^2 \times 5$
   $30 = 2 \times 3 \times 5$

b) $96 = 2^5 \times 3$
   $120 = 2^3 \times 3 \times 5$

c) $216 = 2^3 \times 3^3$
   $144 = 2^4 \times 3^2$

d) $120 = 2^3 \times 3 \times 5$
   $132 = 2^2 \times 3 \times 11$
   $315 = 3^2 \times 5 \times 7$

e) $24 = 2^3 \times 3$
   $105 = 3 \times 5 \times 7$
   $22 = 2 \times 11$

f) $x = a^3 \times b^3 \times c^3$
   $y = a^4 \times b^3 \times c$
   $z = a^2 \times b^2 \times c^2$
   where a, b and c are prime

g) $x = a^3 \times b^3 \times c^3$
   $y = a^4 \times b^3 \times c$
   $z = a^2 \times b^2 \times c^2$
   where a, b and c are integers (i.e. may be prime or not prime)

V In *Example 4,* students are being asked to extend the concept of finding the highest common factor, firstly with two numbers and then extending to three numbers, including when indices have been used.

- In part a), students need to recognise that $20 = 2 \times 2 \times 5$.
- In part b), the index values of two have been increased.
- In part c), students extend to two factors having index values greater than one.
- In part d), there are three expressions to consider.
- In part e), there is no common factor other than one.

When thinking about the algebraic examples in parts f) and g), students are forced to examine the structure of such expressions, reminding them that the factors involved must be prime and, therefore, we simply do not know what the answer is for part g).

D For part g), students may need to be convinced that we do not know the answer to this by finding values for *a*, *b* and *c* so that the highest common factor is not $a^2 \times b^2 \times c$.

One example would be $a = 5, b = 4, c = 2$. Why is this the case?

Find the highest common factor of a pair of numbers when their prime factors are shown in a Venn diagram.

Example 5:

Prime factors of 1 008   **Prime factors of 32 340**
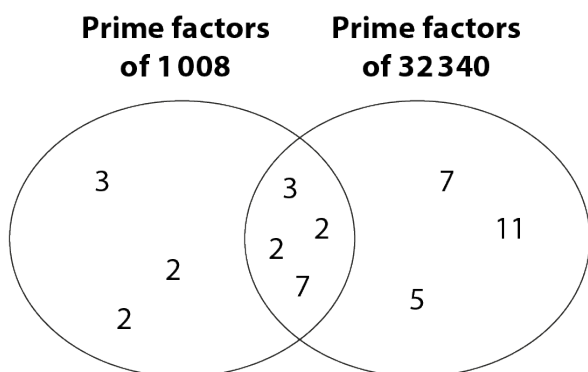
a) (i) *Use the Venn diagram to write 1 008 and 32 340 as products of their prime factors.*

R It is important that students are familiar with a range of different representations and understand that once products are expressed as prime factors, these can be shown clearly in a Venn diagram.

Students should explore how the intersection of sets in a Venn diagram clearly shows all common factors, which can then be used to find the highest common factor.

V
- In part a), students are given a completed Venn diagram and asked to use the information shown to write two numbers as products of prime factors. By doing this, students are demonstrating that they know what the Venn diagram is showing.

(ii) Use the Venn diagram to find the highest common factor of 1008 and 32340.

b) (i) $1575 = 3^2 \times 5^2 \times 7$

$2310 = 2 \times 3 \times 5 \times 7 \times 11$

Show the prime factors of 1575 and 2310 in a Venn diagram.

(ii) Use this Venn diagram to find the highest common factor of 1575 and 2310.

c) (i) Show the prime factors of 165 and 385 in a Venn diagram.

(ii) Use this Venn diagram to find the highest common factor of 165 and 385.

d) Use a Venn diagram to find the highest common factor of 150, 60 and 138.

Students then need to find the highest common factor using the product of the common prime factors shown in the intersection.

- In part b), students are given the numbers in prime factorisation form, but not given the Venn diagram – they must construct it from the information provided and then go on to find the highest common factor.
- Part c) progresses to ask students to draw a Venn diagram from scratch and then use it to find the highest common factor.
- Part d) asks students to find the highest common factor of three numbers using a Venn diagram.

This scaffolding should support students to ensure that all can progress together through each step as a class.

PD How can you assess students' understanding of Venn diagrams? Which familiar concepts could you draw upon to recap the use of Venn diagrams to classify elements?

Explore the most efficient methods of finding the highest common factor.

Example 6:

a) How many different ways can you find to calculate the highest common factor of 50 and 54?

b) Which is the most efficient method, and why?

c) Consider each of your methods. For each method, write a question for which that method is the most efficient way of finding the highest common factor.

d) Can you generalise situations when each method is most efficient?

V Students will benefit from being given time to explore different methods, such as listing factors, writing as product of prime factors or using Venn diagrams.

By considering each method in turn and discussing when it is likely to be most efficient, students will develop a deeper understanding of when it is most appropriate to select specific methods.

Listing factors is likely to be most efficient when the numbers are small.

Writing as a product of prime factors is likely to be most efficient if the numbers are large and there are no repeated factors (where students might experience misconceptions with indices).

Venn diagrams are likely to be most efficient if the numbers are large and there are lots of repeated factors, as these are easily seen visually on the diagram.

Example 7: *Find the highest common factor of these sets of numbers using the most efficient method.* a) *24 and 36* b) *48 and 150* c) *240 and 360* d) ab^2c, a^2c^3 and ab^3c^5	**PD** *Example 7* has been written to encourage students to reflect on which method to select. How will you manage activities like this in your classroom? Is this a task where selected students could come to the board and demonstrate their chosen method to the class, justifying why they have selected it? Could students make predictions and reason about the highest common factor for parts b) and c), based on connections to part a), before calculating it?
Example 8: *The highest common factor of x and y is 63. Which of the following could x and y be?* a) *126 and 189* b) *315 and 126* c) *252 and 126* d) *63 and 441*	**V** *Example 8* is a multiple-choice question with more than one correct answer. Part a) is possibly the most obvious option, but parts b) and d) are also correct. Part c) has a highest common factor of 126 as both numbers have an additional common factor of two. This type of question may provide an opportunity for students to realise and discuss the fact that multiple pairs of different numbers can share the same highest common factor.
Use prime factorisation to solve problems involving the highest common factor. *Example 9:* a) *What is the smallest expression that y can take if $x = a \times b^2 \times c^3$ and the highest common factor of x and y is b^2c?* b) *What is the smallest expression that y can take if $x = c \times d^2 \times e$ and the highest common factor of x and y is def?* c) *If the highest common factor of x and y is ab, how many different pairs of expressions for x and y are there?*	Problems such as those in *Example 9* provide opportunities for students to practise their understanding of a concept (i.e. intelligent practice rather than mechanical repetition) and to focus on relationships, not just on the procedure. **V** An important element of fluency is the ability to work flexibly with a concept. In this example, students are given the highest common factor and asked to find expressions for which it is true. Students could discuss the different methods explored previously in this key idea to help support their reasoning.

Weblinks

[1] NCETM primary mastery professional development materials
https://www.ncetm.org.uk/resources/50639

[2] NCETM primary assessment materials
https://www.ncetm.org.uk/resources/46689

[3] Standards & Testing Agency past mathematics papers
https://www.gov.uk/government/collections/national-curriculum-assessments-practice-materials#key-stage-2-past-papers