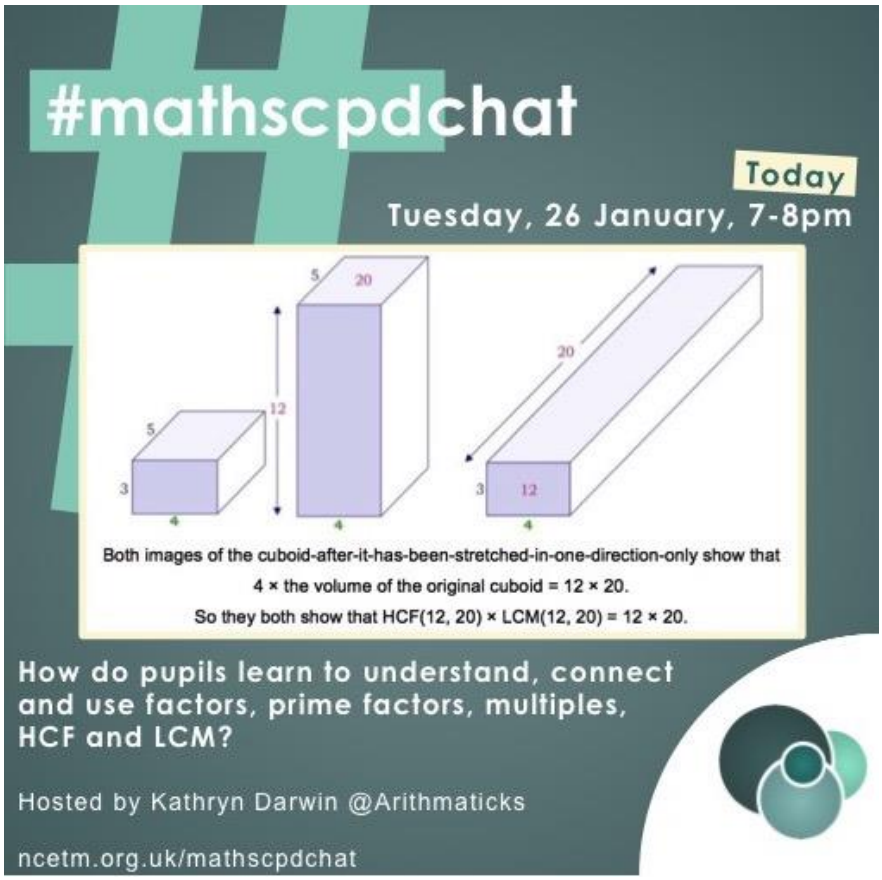
#mathscpdchat 26 January 2021

How do pupils learn to understand, connect and use factors, multiples, HCF and LCM?
Hosted by Kathryn Darwin

This is a brief summary of the discussion – to see all the tweets, follow the hashtag ***#mathscpdchat*** *in Twitter*



Some of the areas where discussion focused were:

ways in which teachers arrange for students to meet multiples and factors for the first time:

- **pupils move objects into groups** (possibly on a whiteboard) … for example, they might arrange four discs in a column, then replicate the column twice to make an array consisting of three columns of four discs;

- **using Cuisenaire® rods** … pupils manipulating real rods or working in an online virtual Cuisenaire® environment such as that at NRICH (link provided below) … placing rods alongside each other to show factors … pupils explore factors and multiples by looking for identical (of same colour and length) rods that can be placed end-to-end to fit exactly the length of a longer rod or of several longer rods (e.g. three white rods fit the length of one green rod, and three green rods, or nine white rods, fit the length of one blue rod … showing, for example, that 9 is a multiple of 3, 3 is a factor of 9, and three 3s are the same as nine 1s);
- there was some **discussion about which idea, that of a multiple or of a factor, teachers should aim for pupils to 'meet'/grasp/understand first** … teachers pointed out: (1) that multiplication creates multiples, division 'undoes' multiplication and reveals factors and (2) that finding multiples is about building-up numbers, and factorising is about breaking-down numbers … while (1) and (2) lead some teachers to believe that 'multiples will always come first via skip counting' (e.g. counting in 2s or 5s or 10s etc.), other teachers see the ideas of multiple and factor as inseparable, for example in seeing (perhaps when moving about and comparing Cuisenaire® rods) that 2 × 3 = 6 a pupil sees all at once that 6 is a multiple of 2 and of 3, and that 2 and 3 are factors of 6;
- there was some discussion about **vocabulary** … about when pupils should first encounter the words 'factor' and 'multiple' … that **'students often get the meanings of the two words muddled'** … some teachers believe that because the words 'multiply' and 'multiple' are so similar it is easier for pupils to understand what a multiple is than to understand what a factor is, and therefore pupils should know what a multiple is before they know what a factor is;
- some teachers use a **grid of rectangles to represent factors and multiples, and focus on how they are related,** in the following way … the rectangles are numbered, 1, 2, 3, 4, … along the top of the grid and down the left-hand side … rectangles going along each row are then coloured in to show multiples (according to the numbers above the rectangles at the top of the grid) of the number at the left-hand end of the row … the coloured-in rectangles (going down each column) then show the factors of the numbers at the top of the grid … 'great visualisation of the relationship between the two';

manipulatives and representations that teachers have found to be particularly helpful in the 'initial' stages of learning about multiples and factors:

- **given a rectangle of a given area thinking about the options for integer side lengths** … choosing dots on a whiteboard square-grid to be corner-dots … making shapes on a geoboard;
- many teachers use **arrays** (of, for example, counters or squares on squared paper) to 'show' pupils (they believe for the first time) examples of what they mean by a 'multiple' … e.g. five rows of three counters to represent a multiple (× 5) of three and a multiple (×

3) of five … then they move on to representing factors as the side-lengths of rectangles with given areas … **'at primary level arrays would definitely be a precursor to an area model'**;

- **using Cuisenaire® rods** … a teacher pointed out that while pupils investigate multiples and factors as described above, they can (may for themselves) see, and make examples that show, that multiplication is distributive over addition … e.g. by seeing a rectangle made by placing three red and three white rods together, as 'three lots of (one red rod + one white rod)' and at the same time as 'three red rods + three white rods' … they may, of course, notice many other facts about, and relationships between, numbers;
- some teachers like pupils to use **Numicon** pieces to explore multiples … pupils seeing the commutativity of multiplication using various Numicon pieces … an image was tweeted that prompted discussion about 4 × 4 = 2 × 8 = 8 × 2;
- some teachers use **Primitives at ptolemy.co.uk** (link provided below) to help pupils appreciate the commutativity of multiplication … e.g. seeing 6 as 2 lots of 3, or 3 lots of 2;
- **dominoes** can be used effectively to represent some number relationships … for example seeing that 7 × 3 = 5 × 3 + 2 × 3 by placing three 5 | 2 dominoes ('portrait'-oriented) side by side … online dominoes manipulatives are useful (links provided below);
- pupils enjoy drawing **'rainbow diagrams'** to show factor pairs … the numbers in each pair are connected by a rainbow arc of a different colour;
- images were tweeted that show how looking at **strips of coloured squares** can prompt a student's awareness that all **the common factors of two numbers are necessarily factors of their difference and of their sum** … students might use this knowledge to facilitate a search for the HCF of two large numbers with a relatively small difference;
- images were tweeted that show how looking at, and **rearranging, squares that tile a rectangle** can prompt students to reason to examples of, and then conjecture that generally, **HCF[a,b] × LCM[a, b] = a × b**;

whether students' ability to understand and work with factors and multiples depends on their being able instantly to recall multiplication facts:

- **some teachers believe that it does** … some teachers commented that 'having number sense' is also important … that understanding multiples and factors is one of the 'foundational blocks' for understanding multiplication tables … **'these things interact'** … teachers commented that being able effectively to use multiples and factors 'are great ways to celebrate fluency with times-tables facts';
- **the game *SticknSplit*** was mentioned by several people as an effective aid in developing 'number sense' … a current research project in which students' learning as a result of playing *SticknSplit* is explored was also mentioned (links provided below);

- that pupils **knowing and being able to use factor-product relationships** (e.g. knowing that 1/6 of 54 is 9 implies that 1/9 of 54 is 6) is more useful to them than merely being able to recall isolated multiplication facts;
- some teachers commented that they themselves **learnt their times tables by rote learning** and by being subjected to 'quick fire' questioning … that it is 'dispiriting' when an A level student cannot instantly recall what number 7 × 6 equals;
- that encouraging pupils to **use 'linked facts' to find multiplication facts** (e.g. working out 5 × 8 as 10 × 4) supports both the development of 'number sense' and learning about multiples and factors;
- that **knowing times tables is not necessary in order to find factors in algebraic expressions** such as a^3b^5 … that 'it's helpful to think of algebra as a foundation of number, rather than the other way round' … that moving both ways between factorising numerical and algebraic expressions supports learning … e.g. moving between finding HCF[738, 750] = 6 in order to factorise 738 + 750 = 6(123 + 125) and finding HCF[a^3b^5, a^2b^7] = a^2b^5 in order to factorise $a^3b^5 + a^2b^7 = a^2b^5(a + b^2)$ … being careful (unlike some textbook writers) to distinguish between the commands to 'factorise' and to 'factorise fully' numerical or algebraic expressions … teachers gave examples as reminders that students should know that common factors may not be integers … e.g. x/2 +3/2 = ½(x + 3);

ways of enabling pupils to avoid confusing the meanings of 'multiple' and 'factor':

- there was much discussion about **whether it is more effective to focus on multiples and factors together at the same time, or to focus on one of them at a time**, and, if so, in what order … many teachers focus on each idea separately before working with them together, usually looking at multiples before factors … contributors to the chat did not appear to reach agreement about this other than that it is better to teach well than to teach unthinkingly however the teaching 'content' is organised … there was a long sub-discussion about when it is advisable to separate in time students' learning about other mathematical ideas that students often confuse (such as area/perimeter and mean/median) and about whether relevant research evidence exists;
- that, whether or not teachers focus on factors and multiples together, enabling students not to confuse the meanings of words for frequently confused mathematical ideas may be enhanced by **exploring the history/derivation (etymology) of those words**, and of words the mathematical meanings of which differ from their 'ordinary life' meanings … that it can be helpful to look at what is the same and what is different about mathematical ideas that students often confuse;
- there was some discussion about the fact that **confusion between the meanings of 'multiple' and 'factor' often arises for the first time only when students start to work with the concepts of Highest Common Factor (HCF) and Lowest Common**

Multiple (LCM) … the word 'lowest' in 'LCM' may suggest 'breaking down' and thus division and thus factor … someone pointed out that **GCSE examiners** report that when the HCF is the correct response to an exam question students often give both the HCF and the LCM, and so gain no marks;

- teachers agreed that it is effective to **wait until students have learned quite a lot about a mathematical idea** before expecting them to know and use the conventional mathematical name for/of it;

how teachers introduce prime numbers:

- some teachers 'facilitate the discovery' by pupils that when the number of objects that they are trying to arrange in a rectangular array is a prime number **only one kind of rectangular array (n × 1) can be formed** …some teachers prompt pupils to use **Cuisenaire® rods** to look for numbers of unit (white) rods that cannot be arranged to form rectangles;
- some pupils explore the **sharing of different numbers of sweets between different numbers of friends** … that is, they see for themselves that for particular numbers of sweets, the sweets cannot be shared equally between any number of friends that is less than the number of sweets … pupils are then informed that such numbers are conventionally called 'prime' numbers;
- there was some discussion about the history behind the decision to **redefine 'prime number' so that the number 1 is excluded**;
- several teachers mentioned **divisibility tests**, giving examples;
- once students have developed a good understanding of multiples and factors, many teachers prompt them to use the ***Sieve of Eratosthenes*** to identify and explore prime numbers;
- **prime factorisation is 'a fave topic' for many teachers** … for example some teachers like 'wondering out loud' in front of students … for example, "I wonder if there is a general relationship between the prime-factor decomposition of a number and the total number of factors that it has" … teachers tweeted images showing links between factors of numbers by representing all the factors of one number on a 2-D or 3-D lattice, with each factor at a different node of the lattice … a substantial discussion developed about properties and implications of such lattice diagrams;

how teachers try to facilitate identification of the Highest Common Factor (HCF) and Lowest Common Multiple (LCM):

- some teachers show their students how to draw and compare **'prime factor trees' or 'prime factor ladder diagrams'** for two or more numbers in order to reveal the prime factors of each number, and so be able to pick out the common factors, and therefore the HCF, of the numbers for which the trees/ladders were drawn … some teachers avoid

using tree diagrams in this context because they are used elsewhere in mathematics (for example in representations to aid the calculation of probabilities);

- some teachers use **Venn diagrams** … the elements of any set are the factors of a particular number (which is the name of the set) … if the elements are prime factors each prime factor is repeated as many times as it appears in the prime-factor-decomposition of the number … so the common factors of numbers lie in the intersection of their sets … the HCF is the product of the factors in the intersection, and the LCM can be deduced from the numbers in the union of the sets … a teacher tweeted diagrams to show how he helps students use Venn diagrams … he slides images of the two separate sets containing prime factors towards each other until they overlap … and instantly the two representations of each instance of a common prime factor (now within the intersection of the sets) 'collapse' to become one number;
- many teachers avoid using both Venn diagrams and tree (or ladder) diagrams … instead they prompt students to **write numbers as (horizontally written) products of their prime factors** … that is, pupils first write a number as the product of any two of its factors, each such factor is then replaced by the product of two of its factors (the whole new product of four numbers being reproduced on the line below) … this process is repeated until the student has obtained the prime-number-decomposition of the number … the HCF and LCM of a set of numbers can be picked out by comparing the horizontally-written products of their prime factors;
- there was a short discussion about how when students have followed **a long step-by-step procedure** to obtain a result, they are often in a good position to appreciate the value of getting to understand and use (or even devise for themselves) **a shorter (but harder to understand) alternative method**;
- a teacher pointed out that the **ability of pupils in Y5/6** to add/subtract fractions with different denominators **can be built on when they start to learn about the LCM of numbers**;
- a by-product of the discussion about finding the HCF and LCM of numbers was a **discussion about the use of 0 and 1 as identity operators** … that adding/subtracting 0 and multiplying/dividing by 1 are both identity operations … the effects of operating with 0 and 1 are highlighted in the NCETM Primary Mastery PD materials (link provided below);

how teachers introduce students to the 'Fundamental Theorem of Arithmetic' … that every positive integer (except 1) can be represented in exactly one way as a product of one or more primes:

- teachers challenge students to explore ways of writing any number as a product of integers **in as many different ways as possible**, compare their products, then break

down each of their different products into products of prime numbers and compare them again;

- some teachers challenge students to make prime factor trees for a number that has plenty of factors (such as 120) and to **find a route to the prime factors that no-one else has found**;

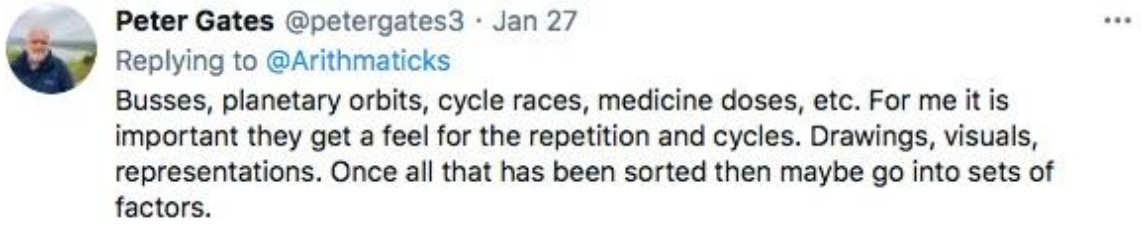
what teachers regard as the best prompt, resource or idea to support learning about multiples, factors and prime numbers:

- many teachers like using @mathsbot's **virtual prime factor tiles** (link provided below);
- many teachers like using the **Sieve of Eratosthenes** for finding prime numbers;
- some **games of skill from NRICH** were provided as examples of 'favourite tasks' to support the development of fluency in finding factors (links provided below);
- that the **LCM, HCF inquiry prompts** at inquirymaths.org 'are great for finding out what pupils already know and for motivating pupils to find out more'.

In what follows, click on any screenshot of a tweet to go to that actual tweet on Twitter.

This is a part of a conversation about the manipulatives and representations that teachers use when pupils are in the early stages of learning about factors and multiples. The conversation was generated by this tweet from Kathryn Darwin:

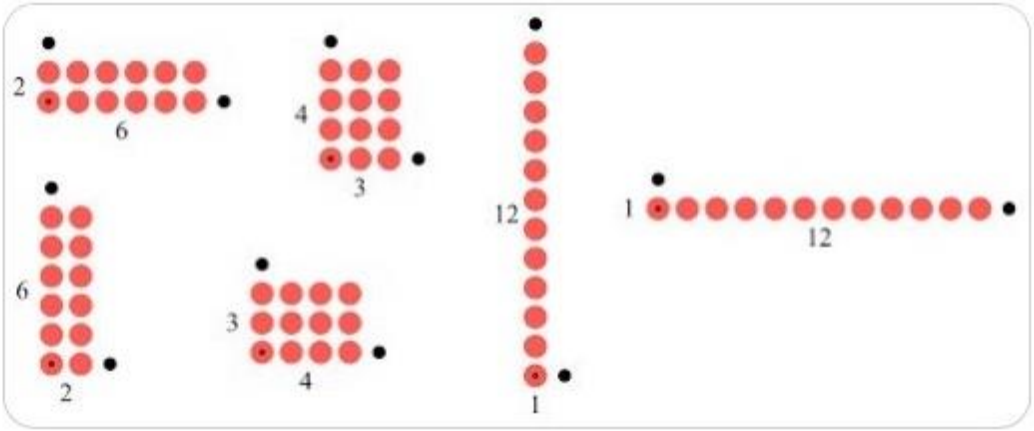
and included these from Peter Gates, Tazreen Tershanah and Tom Oakley:



Tazreen Tershanah @tershanah · Jan 26

Replying to @Arithmaticks and @blatherwick_sam

#mathscpdchat At primary level arrays would definitely be a precursor to an area model and could be used to bridge the gap between understanding the difference between factors and multiples



Tom Oakley @ThatMathsMan · 21h

Replying to @blatherwick_sam and @Arithmaticks

Spatial reasoning is important for developing number sense. As chn, our experience of geometry comes before our experience of number and we use spatial reasoning to understand any pictorial representation, so I think it's important to link arrays etc with factors, multiples etc.

these from Martyn Yeo, Kathryn Darwin and Sharon Malley:

Martyn @martynyeouk · 16h

Replying to @Arithmaticks

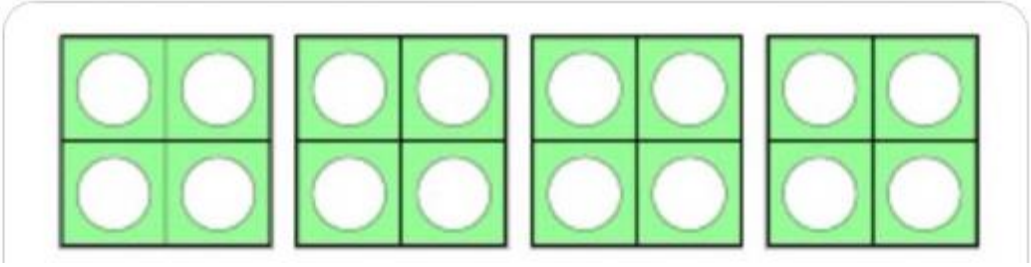
Always love a bit of @Numicon to show multiples #mathscpdchat

Kathryn MCCT @Arithmaticks · 16h

Can you show us anything to help us visualise this? #mathscpdchat

Martyn @martynyeouk · 16h

Struggling a bit on my phone - but multiples of 4 started here....
#mathscpdchat



Sharon Malley @mathsmumof2 · Jan 26

Replying to @martynyeouk @Arithmaticks and @Numicon

And you could also show how it is equal to 2 x 8 and 8 x 2 as well using the tiles #mathscpdchat

these from Sam Blatherwick, theperfectlanguage, Atul Rana and Mary Pardoe:

Sam Blatherwick @blatherwick_sam · 16h

Replying to @Arithmaticks

For factors, given a rectangle of a specific area, thinking about the options for integer side lengths. You can then see how those rectangles can be split further. #mathscpdchat

theperfectlanguage @theperfectlang1 · 16h

Replying to @Arithmaticks

Rectangles are good for showing the factor pairs of numbers using their areas. They can be made from multi-link, drawn or created on a geoboard so are a low impact in terms of resourcing.

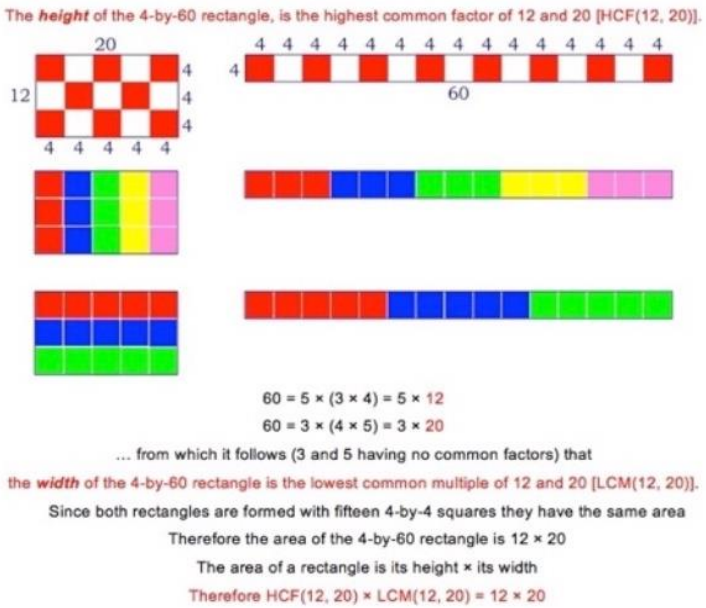
Atul Rana @atulrana · 16h

This is one of my most favourite resources for primeness, factorisation etc. I would have got tutees to move these manually first as dots on the whiteboard first before showing them this animation #MathsCPDchat datapointed.net/visualizations...

Mary Pardoe @PardoeMary · 18h

Replying to @blatherwick_sam and @Arithmaticks

Yes. Looking at rectangles can take students' reasoning quite a way!

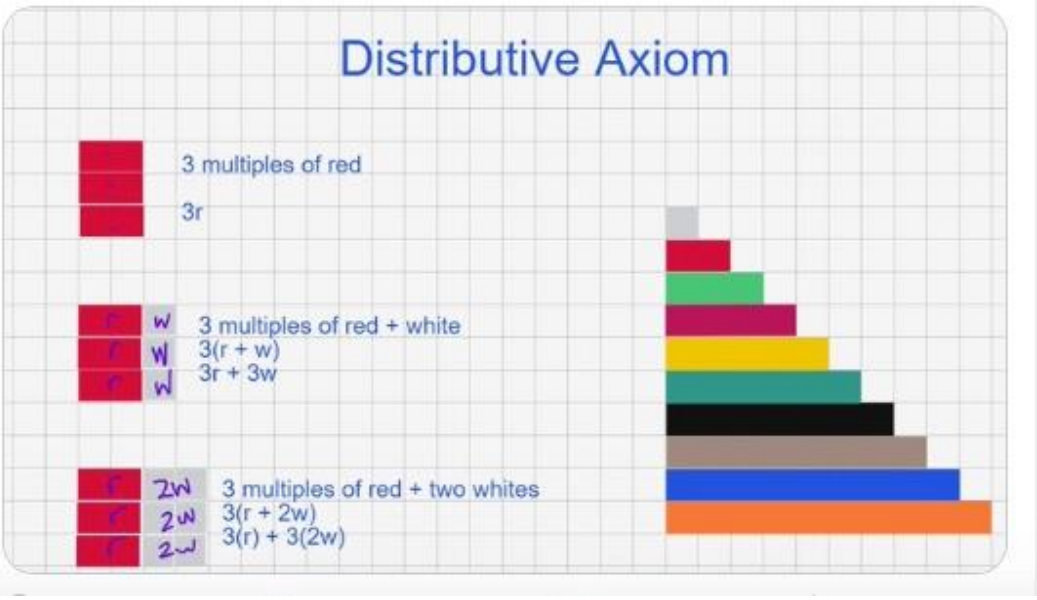


and these from Seamus Murphy and Gemma Scott:

Atul Rana @atulrana · 20h

Replying to @atulrana @Arithmaticks and @blatherwick_sam

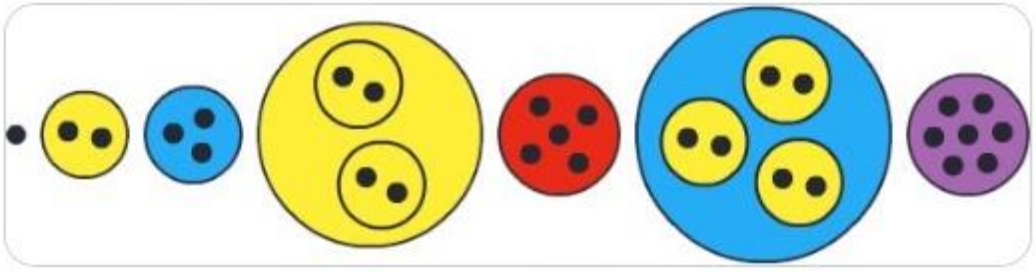
I would introduce them to factorisation with 'letters' only after I've covered the distributive axiom typically with cuisenaire rods #MathsCPDchat



Tom Francome @TFrancome · 16h

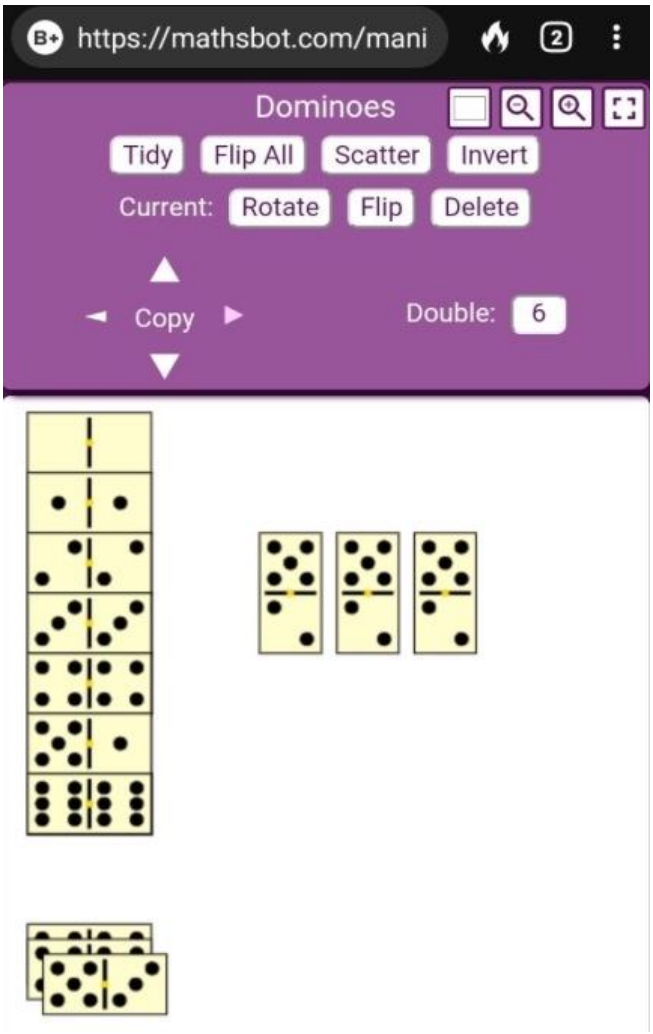
Replying to @Arithmaticks

@alecmce Primitives is one of my favourites. Particularly how you can change the ordering and see two lots of 3 or three lots of 2.
ptolemy.co.uk/2007/09/19/pri...



Tom Oakley @ThatMathsMan · 19h

You can show, for example that 7 × 3 is (5 × 3)+(2 × 3). Particularly easy to show on mathsbot etc.
#mathscpdchat



(to read the discussion sequence generated by any tweet look at the ‘replies’ to that tweet)

Among the links shared were:

Prime Factor Tiles which is one of the many manipulatives at mathsbot.com. Users work in a space with coloured tiles, each of which represents (is labelled with) a prime number. It was shared by Tom Francome

Dominoes which is another of the many manipulatives at mathsbot.com. Users can work in a space with dominoes. It was shared by Tom Oakley

Cuisenaire Environment which is an application at NRICH. Users can drag 2-D representations of Cuisenaire® rods into the working space, and rotate, drag and delete them. It was shared by Mary Pardoe

Factors and Factorisation of Numbers which is a YouTube video by Peter Mattock which is designed to support teachers of mathematics in helping learners make sense of factors and factorisation of numbers using Cuisenaire® rods. It was shared by Peter Mattock

Polypad which is an *mathigon* application. Any number of a variety of coloured polygons can be dragged into the working space and manipulated. There are also options to work with tangram pieces, polyominoes, Penrose tiles, and other kinds of tile. It was shared by MIGU3L_HH

Primitives which is an application from ptolemy.co.uk in which numbers are presented as nested sets of small black dots. When a number has many different prime factors, the factors can be rearranged to offer different images of the same number. The old Flash versions of the applications no longer function, but there was a discussion during the chat between Tom Francome and Alec McEachran, the creator, about the possibility of recoding for the modern web. It was shared by Tom Francome

Datapointed factor animation which is an application that represents the factors of numbers as attractive images of small coloured discs arranged appropriately and significantly. The numbers whose factors are represented increase and increase … . It was shared by Atul Rana

Bitpaper which is an application that enables the user to write and draw on a whiteboard. The user can work and communicate online in real time, and share the screen with websites, apps, videos and more. You can also save, add rearrange and duplicate pages that you can re-use time and time again. It was shared by Atul Rana

Stick and Split which is a game from Sunflower Learning Limited that was designed to help people learn multiplication and division facts. The application is a deceptively simple game that was seven years in development. It was shared by Sharon Malley

Collaborative Studentship: A Game-based Approach to Improving the Learning of Arithmetic which is a link to where you can apply for a Doctoral Studentship to commence in October 2021. Successful applicants will work on a project based in the Mathematics Education Centre at Loughborough University. The aim of the project is to identify how children's learning of arithmetic is developed through the *Stick and Split* game (link above). It was shared by Tom Francome

Prime Factors, HCF and LCM which is a tool at mathsbot.com. Users can enter, or let the application generate, and hide, numbers in a four-column table with column-headings x, y,

HCF(x, y) and LCM(x, y) and with whatever number of rows the user selects. It was shared by Charlotte Hawthorne

HCF and LCM inquiry which is one of the many prompts to generate mathematical activity and learning that are clearly presented on the inquirymaths.org. website. This one was created collaboratively by the Mathematics team at The Brittons Academy (Rainham, Essex). It was shared by Helen Hindle

Factor Track which is a game of skill at NRICH. The aim is to go round a track consisting of squares, many of which contain a single positive whole number, in the least number of moves. You have to move to a square containing a factor of the number in the square that you are on. It was shared by Tazreen Tershanah

All Secondary Factors and Multiples which is the page on the NRICH website from which you can reach every NRICH task, game or application intended for secondary-school students that involves factors and/or multiples Us. It was shared by Mary Pardoe

Stars which is an application at NRICH. Users draw stars by joining (with straight line-segments) 'points' (the number of which the user selects) that are spaced at equal distances around a circle. Users can be challenged to find a general relationship between the number of 'points' and the number of steps needed to visit every 'point'. It was shared by Mary Pardoe

Charlie's Delightful Machine which is a well-known application at NRICH. Users have to work out how to switch on some lights by entering numbers in a box and deducing rules that control the switches. It was shared by Mary Pardoe

Fingerprints of a Number which is a blog by Sam Blatherwick that was inspired by a problem (from the *Stanford Mathematics Problem Book* by George Pólya and Jeremy Kilpatrick) which required you to find all the ways in which 72 can be written as the product of three of its factors. Sam looks at interesting approaches to finding factors of numbers. It was shared by Sam Blatherwick

Fingerprints of a Number 2 which is a follow-up to the above blog by Sam Blatherwick which he wrote on the day after this #mathscpdchat as a result of discussions during the chat. It was shared by Sam Blatherwick

Prime Factor Form which is a blog by Helen Konstantine from which you can freely download many attractively presented tasks involving prime factors which were created, and are generously shared, by Helen. It was shared by Helen Konstantine

Fundamental Theorem of Arithmetic which is a definition/explanation from *Wolfram MathWorld*. It was shared by Sharon Malley

Primary Mastery Professional Development are NCETM materials designed to assist teachers in their professional development and enable them to teach mathematics for mastery with confidence. It was shared by Alison Hopper

GCSE Mathematics - small things make a big difference which is a PDF document from AQA in which examiners highlight where marks are unnecessarily lost in mathematics GCSE exams. It includes a reference to instances where students confuse the meanings of HCF and LCM. It was shared by Tessmaths