National Centre for Excellence in the Teaching of Mathematics

Professional development resource

Unit 2 - Understanding and identifying proportional contexts
Lesson 2f: Using the DNL to solve ratio (and non-ratio) tasks

Lesson summary

In this lesson we return to the contexts and tasks of Lesson 2e, and consider how or whether the double number line (DNL) is useful for solving these tasks, eg by working 'along the lines' and 'between the lines'. We provide opportunities to work with more awkward numbers, which might help some students adopt a more formal 'between the lines' *scaling* approach.

Focus of student learning

In this lesson we consider how the DNL can be used to solve ratio (and non-ratio) tasks.

- How readily do students adopt 'along the lines' and 'between the lines' approaches?
- Are students aware that the effectiveness of these approaches can vary with context and input?
- How aware are students of the power of a 'between the lines' approach for ratio tasks?

Lesson preparation

- The file **UNIT-2ef-slides.pdf** contains
  - [various slides for Lesson 2e]
  - slides for Lesson 2f Stage 1
  - slides for Lesson 2f Stage 2
  - a number-relations challenge (for us rather than the students?).

The Lesson

Stage 1.

Here we return to some of the contexts from Lesson 2e. In each of these, 6 maps onto 15*. This time we try to find the image of 20 (and perhaps of other, less salient numbers like 23). We use the DNL to *show* and to *analyse* what is going on and consider how helpful it is for *finding* ways of solving the task, for different contexts.

[*At some point, you might want to consider what happens when 6 ⟶ 15 is changed to 6 ⟶ 14, say.]

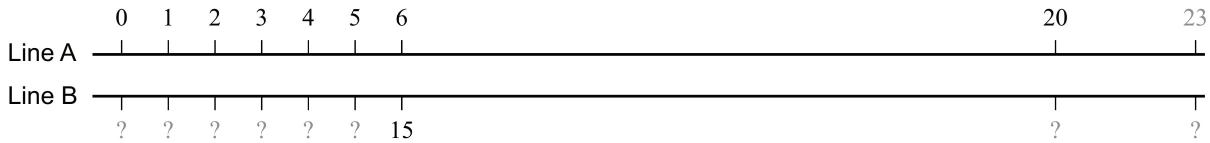
1 a) When Jon was 6 years old, his cousin Lee was 15 years old.

i. When Jon is 20, how old is Lee?

ii. When Jon is 23, how old is Lee?

Think about how we might use the DNL, below, to solve this problem.
(Line A represents Jon's age and Line B represents Lee's age.)

- How helpful is the DNL?
- Are there different ways of solving the problem?



Stage 2.

Here we repeat part a) for some (or all) of the other contexts from Lesson 2e. You might decide not to consider the mapping 23 ⟶ ?, or you might want to choose (or ask students to choose) other mappings.
We finish by focussing on ratio contexts, where the relation is purely multiplicative (task 2).

Choose some of these situations from Lesson 2e [see parts b) to h) below].

- Does the DNL help us (and if so, how) to solve each problem?
- Try to find several ways to solve each problem.

b) i. How many litai are \$20 worth?
ii. How many litai are \$23 worth?

c) i. How many pounds does Meg weigh when she is 20 months old?
ii. How many pounds does Meg weigh when she is 23 months old?

d) i. How long does it take to cook 20 medium size potatoes?
ii. How long does it take to cook 23 medium size potatoes?

e) i. What would a 20 ft length of the sterling silver wire cost?
ii. What would a 23 ft length of the sterling silver wire cost?

f) i. What would Jenny be paid per hour if the £90 job took 20 hours?
ii. What would Jenny be paid per hour if the £90 job took 23 hours?

g) i. What does a 20 cm wide cardboard disc weigh?
ii. What does a 23 cm wide cardboard disc weigh?

h) i. How long is a line on Plan B that is 20 cm long on Plan A?
ii. How long is a line on Plan B that is 23 cm long on Plan A?

2 a) For which of the situations can a mapping (eg 20 ⟶ ?) be solved just by multiplying by a specific number, whatever the input?

b) Make up other 'stories' where this is true.

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Lesson commentary

Stage 1.

The mappings in this lesson can often be solved by working 'along-the-lines' or 'between-the-lines' of the DNL. The lesson should heighten students' awareness that for a ratio context, the 'between-the-lines' approach can be reduced to multiplying (or more precisely to *scaling*) by a single number, albeit not necessarily a whole number.

1 a) This context involves a simple additive relation and so is quite easy to solve, regardless of the input, and without the DNL. However, the DNL is still useful here for *illustrating* and *analysing* the methods used.
For part i. (Jon is 20) an along-the-lines approach can be expressed like this:
6 + 14 = 20, 15 + 14 = 29 (Jon is 14 years older than he was, so Lee is 14 years older than he was);
a between-the-lines approach can be expressed like this:
6 + 9 = 15, 20 + 9 = 29 (Lee is 9 years older than Jon, so when Jon is 20, Lee is 20 + 9).

In general, students prefer to work along rather than between the lines. For a **ratio context**, and an input of 20, there are several effective along-the-lines strategies, for example:

6 + 6 + 6 + one-third-of 6 = 20, 15 + 15 + 15 + one-third-of 15 = 50, or

$6 \times 3\frac{1}{3} = 20$, $15 \times 3\frac{1}{3} = 50$, or

$6 \div 6 = 1$ and $1 \times 20 = 20$, $15 \div 6 = 2.5$ and $2.5 \times 20 = 50$.

Some students might use such strategies on the *How old is Lee?* task, even though they don't fit here.

When the input is 23, an along-the-lines approach is not quite so simple for a ratio context. Thus the purpose of using this input is to encourage students to consider a between-the-lines approach. Such an approach can be applied directly to *any* number on the top line (even for non-ratio relations), in contrast to an along-the-lines approach which has to be tailored to the specific number being mapped.

Stage 2.

Here students get a chance to discover that the nature and effectiveness of along- and between-the-lines strategies may differ for different contexts. Further, the effectiveness of along-the-lines strategies can be highly dependent on the value of the input.

b) Along: eg, $6 \times 3\frac{1}{3} = 20$, $15 \times 3\frac{1}{3} = 50$ [or $6 \times 3\frac{5}{6} = 23$, $15 \times 3\frac{5}{6} = 57\frac{1}{2}$];
but look out for the 'addition strategy': 6+14=20, 15+14=29 [or 6+17=23, 15+17=32].
Between: multiply 2.5 by 20 [or by 23];
but look out for the 'addition strategy': 6+9=15, 20+9=29 [or 6+9=15, 23+9=32].

c) As we don't have actual data on Meg, we could make estimates based on similar data. Estimates based just on the limited data shown on page 7 of the Lesson 2e resource would be rather unreliable - it is tempting to draw a graph here and continue the curve (essentially an along-the-lines approach).

d) We might want to assume that the cooking time is always 15 minutes.
Or we could make an assumption like 'each extra potato increases the cooking time by 0.2 minutes'. This could lead to the along-the-lines expression 15 + 14×0.2 minutes for 20 potatoes [or 15 + 17×0.2 minutes for 23 potatoes], or the between-the-lines expression 13.8 + 20×0.2 [or 13.8 + 23×0.2]. This model has a similar structure to *Zoom Taxis* from Lesson 2e.

e) This can be solved in the same ways as the $ to litai conversion task in part b). Do students notice this?

f) A complete along-the-lines approach is complex here, although we could move towards the desired result like this: 6×3=18 hours would be paid at a rate of 15÷3 = 5 £ per hour. A between-the-lines approach is much more direct: the rate of pay is 90 divided by 20 [or 90 divided by 23] £ per hour.

g) As with part f), we can use an along-the-lines approach to get near the answer: eg, an 18 cm disc weights $3^2 = 9$ times as much as a 6 cm disc. Again, a between-the-lines approach is much more direct (but perhaps too complex to resolve fully for some classes): the disc weighs $\frac{5}{12} \times 20^2$ g [or $\frac{5}{12} \times 23^2$ g].

h) This can be solved in the same ways as the $ to litai conversion task in part b), but it might attract more 'addition strategy' responses (leading to 20 ⟶ 29 [or 23 ⟶ 32], instead of 20 ⟶ 50 [or 23 ⟶ 57.5]).

2 a) Several of the tasks can be solved using a between-the-lines approach, but only the direct ratio tasks involve a single, between-the-lines multiplier, in this case ×2.5. You could ask, "Is there an efficient calculator method?"
Although 2.5 is a relatively simple rational number, students might find it challenging to discern (and use) a rational number multiplier. This is a key mathematical idea which takes a long time to develop.

b) You could use the 'stories' task as an extension or homework activity, or base a lesson on it. It is also important to explore what happens with different numbers (eg with 6 ⟶ 15 changed to 6 ⟶ 14).

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Adapting the lesson

- The tasks have a strong diagnostic element and should be accessible to most students. However, you need to decide whether to include the input 23 or whether to replace it with a simpler number (eg 21 or even 24) or more complex number (eg 23.5).
- The tasks you use in Stage 2 will depend on which ones you used in Lesson 2e. It is not essential that all the tasks are tackled or that the more challenging tasks [f, g] are solved fully. However, you should try to use several of the ratio tasks [b, e, h] and at least one of the non-ratio tasks [f, g]. The 'indeterminate' tasks [c, d] are less important in this lesson.
- If you don't reach task 2 of Stage 2, try to find time for this in a subsequent lesson.

Suggestions for lesson study

You might want to consider these aspects of students' thinking:

- How readily do students adopt 'along the lines' and 'between the lines' approaches?
- Are students aware that the effectiveness of these approaches can vary with context and input?
- How aware are students of the power of a 'between the lines' approach for ratio tasks?

You might also want to consider pedagogic aspects such as these:

- How did we (and how, if we teach the lesson again, could we find further ways to) elicit (and share) examples of students' thinking?
- How did we (and how, if we teach the lesson again, could we find further ways to) build on and interrogate students' thinking?

Research background to the lesson

The ratio table (right) shows Stage 2 task 2b in abbreviated form:

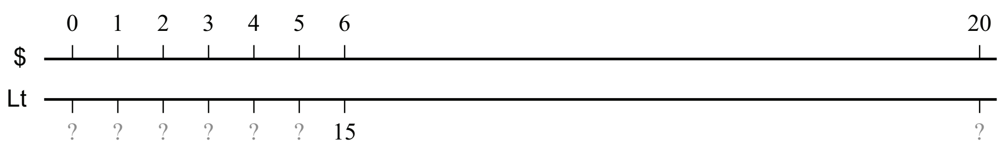
If 6 dollars are worth 15 litai, how many litai are 20 dollars worth?

$	6	20
litai	15	•

There is evidence to suggest that for tasks like these, student prefer (if the numbers are comparable) to form relations between numbers in the same 'measure space', ie they prefer to find a relation between 6 and 20 (eg $6\times3\frac{1}{3}=20$) and apply it to 15 ($15\times3\frac{1}{3}=50$) than to find a relation between 6 and 15 (eg $6\times2.5=15$) and apply it to 20 ($20\times2.5=50$).

In this and the previous lesson, we have used DNLs that go through the same measure space (ie, for the task above, one line represents dollars [and includes the 6 and the 20] and the other represents litai [and includes the 15 and the 50]).[1] This means that (if the numbers are comparable) students will tend to prefer to work 'along the lines' rather than 'between the lines'. Thus students might initially resist using the method of scaling even though it maps every number on the top line onto the number directly below on the other line.

Scaling is an important idea which takes time to develop. It provides a view of multiplication that is quite distinct from those based on the area model or on repeated addition.



1. It would be possible to construct an 'orthogonal' DNL (ie with a 'vertical' line that includes the 6 and the 15 and a second 'vertical' line that includes the 20 and the 50), though it may not be as easy to interpret.