MathsHUBS Resources

National Centre for Excellence in the Teaching of Mathematics

Mastery Professional Development

Fractions

3.9 Multiplying fractions and dividing fractions by a whole number

Teacher guide | Year 6

Teaching point 1:

When a fraction is multiplied by a proper fraction, it makes it smaller. To multiply two fractions, multiply the numerators and multiply the denominators.

Teaching point 2:

When a fraction is divided by a whole number, it makes it smaller. To divide a fraction by a whole number, convert it to an equivalent multiplication.

Teaching point 3:

A more efficient method can be used to divide a fraction by a whole number when the whole number is a factor of the numerator.

2019 pilot

Overview of learning

In this segment children will:

- learn how to multiply pairs of proper fractions
- confirm that the commutativity of multiplication extends to pairs of proper fractions
- review and extend their knowledge that when a number is multiplied by a proper fraction, it is made smaller
- learn how to divide a proper fraction by a whole number
- review the fact that, for example, multiplying by $\frac{1}{3}$ is same as dividing by three.

This focused segment is intentionally kept brief. Multiplication and division of fractions underpins much of the proportional reasoning that children will draw on in the secondary curriculum and, as such, this topic will be revisited in secondary school. Realistically, children will only learn to multiply and divide fractions in an introductory form at primary. However, there are some really foundational ideas in this segment that primary-aged children are capable of grasping, and which will provide a really solid platform for their secondary work.

One such fundamental concept is that multiplying a number by a proper fraction makes it smaller. This conceptual understanding is just as important as being able to perform a procedure for multiplying and dividing fractions. Children were originally introduced to this in relation to multiplying a *whole number* by a proper fraction in segment *3.6 Multiplying whole numbers and fractions,* and this is now extended to multiplying a *fraction* by a proper fraction.

The idea that when we multiply by a proper fraction we are making a number smaller, is really significant because in some branches of higher mathematics, division ceases to be a concept that is used, since any division can be replaced by multiplication (for example, dividing by four becomes multiplying by $\frac{1}{4}$). The notion of multiplication as scaling may help here, since scaling can make things larger or smaller. *Spine 2: Multiplication and Division*, Segment *2.27* also explores this idea in the context of multiplication as scaling. Moving away from thinking that multiplication always makes things bigger is an important awareness to develop in preparation for secondary school.

In this segment, the children learn generalisations that will allow them to multiply pairs of fractions and divide fractions by whole numbers. The generalisations reached are suitable for use in primary schools, but do not necessarily offer the most efficient method. In secondary school, children will be taught to check whether fractions can be simplified before multiplying by cancelling common factors in the numerators and denominators. This is not expected for primary children, but it can be particularly beneficial with larger denominators, e.g.:

$$\frac{2}{{}_{1}\cancel{3}} \times \frac{\cancel{3}^{1}}{5} = \frac{2}{5}$$

$$\frac{{}^{1}\cancel{3}}{{}_{2}\cancel{10}} \times \frac{\cancel{5}^{1}}{\cancel{12}_{4}} = \frac{1}{8}$$

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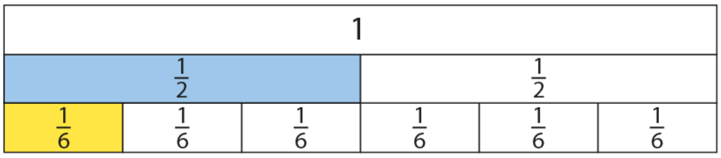
This is beyond the primary programme of study, and therefore there is inevitable over-generalisation in the procedure of multiplying the numerators and multiplying the denominators. As the procedures for multiplying pairs of fractions and dividing fractions by whole numbers are so simple, it is likely that children will quite quickly become fluent in performing them. This is fine – it is unnecessary to make children draw out diagrams when there is a quick procedure which they can use. However, do keep checking that children are 'sense-making' in terms of what they are doing, in particular that they recognise that in these calculations they are making a number smaller.

Through both *Spine 3: Fractions* and *Spine 2: Multiplication and division*, the fact that multiplication expressions can be written with the multiplier and multiplicand in either position has been emphasised (e.g. three lots of $\frac{1}{5}$ can be written as both $\mathbf{3}\times\frac{1}{5}$ and $\frac{1}{5}\times\mathbf{3}$). Confidence in moving between these two ways of writing a multiplication expression is crucial in this segment. Sometimes the *multiplier* is used first, to help reinforce the link between, for example, $\frac{1}{3}\times\frac{1}{2}$ and $\frac{1}{3}$ *of* $\frac{1}{2}$.



However, when progressing to division of fractions, the link between multiplication by a fraction and division by an integer is clearest when the *multiplicand* is written first, e.g. $\frac{1}{2}\times\frac{1}{3}=\frac{1}{2}\div\mathbf{3}$ (so, $\frac{1}{2}$, 'thirded', is the same as $\frac{1}{2}$ divided by three). The method used for division of fractions by a whole number is to convert to an equivalent multiplication, so an understanding of this equation is essential in order for children to understand why they can rewrite $\frac{1}{2}\div\mathbf{3}$ as $\frac{1}{2}\times\frac{1}{3}$ in order to solve it. As a result, you will notice that through the segment, the examples alternate between these two ways of writing multiplication expressions, and it is expected that the children will become confident in doing this also.

The final teaching point focuses on a second approach to dividing fractions. This approach can be used when the numerator is a multiple of the divisor, for example in $\frac{6}{7}\div\mathbf{3}$, six is a multiple of three. Here, children will draw on their understanding of the unitising that has been applied throughout this spine, and the knowledge that $\frac{6}{7}$ is six lots of the unit fraction $\frac{1}{7}$. Six one-sevenths can be split into two equal parts in the same way integers can be. If *6 (ones) ÷ 3 = 2 (ones)*, then *6 (one-sevenths) ÷ 3 = 2 (one-sevenths).*

One consistent model (shown above) is used throughout this segment to illustrate the concepts. During this spine, children have been taught that fractions can be both operators, e.g. $\frac{1}{3}$ *of* a number or measure, and fractions can be numbers with a position on a number line. A linear model of fractions emphasises the similarities in these two uses very clearly, and will provide a good foundation for the secondary curriculum. An area model can also be used to illustrate multiplication and division of fractions, shown here exemplifying $\frac{2}{3}\times\frac{4}{5}=\frac{8}{15}$.

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$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$
$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$
$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$

It is possibly easier to grasp the multiplication of non-unit fractions through this model, for example the 2×4 that produces the numerator of eight, and the 3×5 that produces the denominator of 15. However, the links with the number line, and the understanding of multiplication by a proper fraction as a linear scaling of a number which makes it smaller, are somewhat lost.

The first model is used throughout this segment to illustrate the concepts, but all of the concepts could equally be taught with the second model if preferred. Make a decision within your school about which model you feel more comfortable with, and apply this one model throughout your teaching. Both models have clear links to core multiplication models used elsewhere in these materials (the first to the bar model and the second to the area model), so in either case children will be able to make connections to other learning.

An explanation of the structure of these materials, with guidance on how teachers can use them, is contained in this NCETM podcast: www.ncetm.org.uk/primarympdpodcast. The main message in the podcast is that the materials are principally for professional development purposes. They demonstrate how understanding of concepts can be built through small coherent steps and the application of mathematical representations. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

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Teaching point 1:

When a fraction is multiplied by a proper fraction, it makes it smaller. To multiply two fractions, multiply the numerators and multiply the denominators.

Steps in learning

Guidance | **Representations**

1:1

Guidance

In segment *3.6 Multiplying whole numbers and fractions*, children learnt how to multiply a whole number by a fraction. They learnt that:

- to find $\frac{1}{4}$ of 20, for example, you need to divide 20 into four equal parts and then find one of those parts
  - this can be expressed as both $\frac{1}{4} \times 20$ and $20 \times \frac{1}{4}$
- when any whole number is multiplied by a proper fraction, such as 20 multiplied by $\frac{1}{4}$, the whole number becomes smaller.

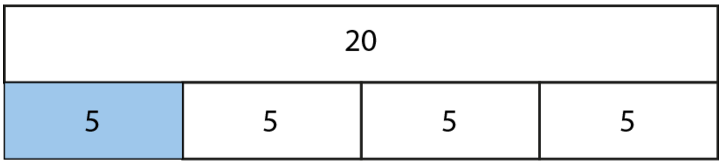
In this segment, children will move on to exploring how this learning extends to multiplying two proper fractions.

You might like to start by looking at halves of the denominator of unit fractions, displaying the first image in the second series of images opposite. From this, it is possible to see that $\frac{1}{4}$ of one is $\frac{1}{4}$. In equation form, this can be written as $\frac{1}{4} \times 1 = \frac{1}{4}$ and $1 \times \frac{1}{4} = \frac{1}{4}$.

Children know that they can find a fraction of a whole number, but how can they extend this to finding a fraction of a fraction? Pose a question such as: *'If one of the quarters is made half as long (represented by the question mark), what fraction of the whole will one*

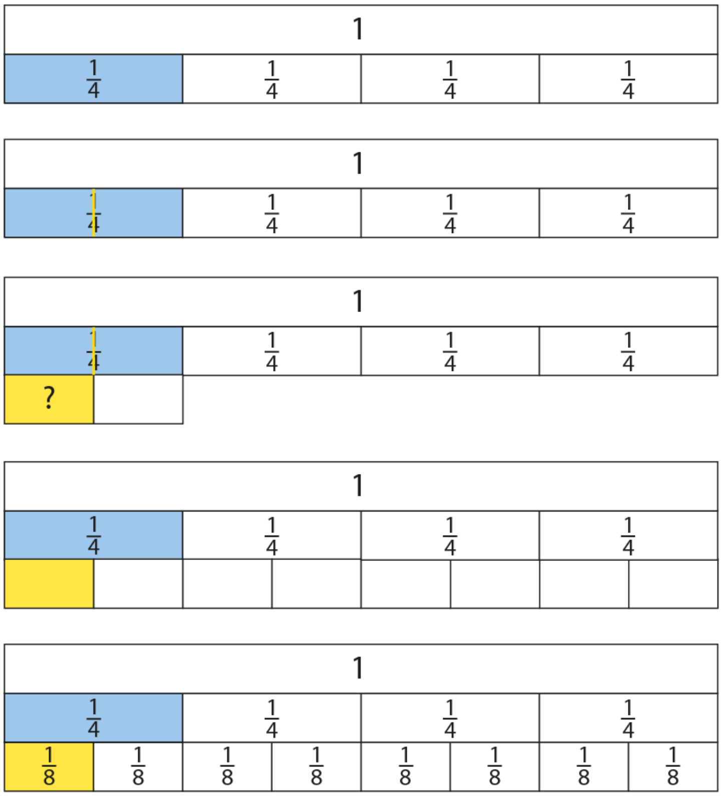
Representations

Multiplying a whole number by a fraction:



Multiplying two proper fractions:

'If one of the quarters is made half as long (represented by the question mark), what fraction of the whole will one of the new equal parts be?'



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	of the new equal parts be?'. This can be linked to previous work on finding the whole from a part. Discuss that there would be eight equal parts of this size in the whole, so half of $\frac{1}{4}$ is $\frac{1}{8}$ ($\frac{1}{2}$ of $\frac{1}{4} = \frac{1}{8}$). In equation form, this can be written as: • $\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$ (one-half of $\frac{1}{4}$ is equal to $\frac{1}{8}$) and • $\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$ ($\frac{1}{4}$, halved, is equal to $\frac{1}{8}$). In both of these equations one-quarter is our multiplicand (starting quantity) and one-half is our multiplier (we are halving our multiplicand). Vary your wording as you read out equations, using both the 'conceptual' words suggested above, as well as symbolic language of 'times' and 'multiplied by'.	
1:2	Explore halves of other unit fractions using the same linear model. You could use the models on the next page and ask: *'If one of the thirds is made half as long, what fraction of the whole will one of the new equal parts be?'*. There would be six equal parts of this size in the whole, so one-half of $\frac{1}{3}$ is $\frac{1}{6}$ ($\frac{1}{2}$ of $\frac{1}{3} = \frac{1}{6}$). In equation form, this can be written as • $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$ (*'One-half of* $\frac{1}{3}$ *is* $\frac{1}{6}$*.'*) and • $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$ ('$\frac{1}{3}$*, halved, is* $\frac{1}{6}$*.'*). At this stage, all of the focus of the	

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discussion should be on sense-making. Children may notice that the denominators have been multiplied, but don't rush to generalise at this time. Instead, discuss why it is that, if $\frac{1}{3}$ is halved, it results in $\frac{1}{6}$, aiming to elicit the following observation: *'There were three equal parts in the whole, each of the three parts was halved, so now we have six equal parts in the whole.'* In the halving of each part, *double the number of parts* were created, so it makes sense that the denominator has doubled, too.

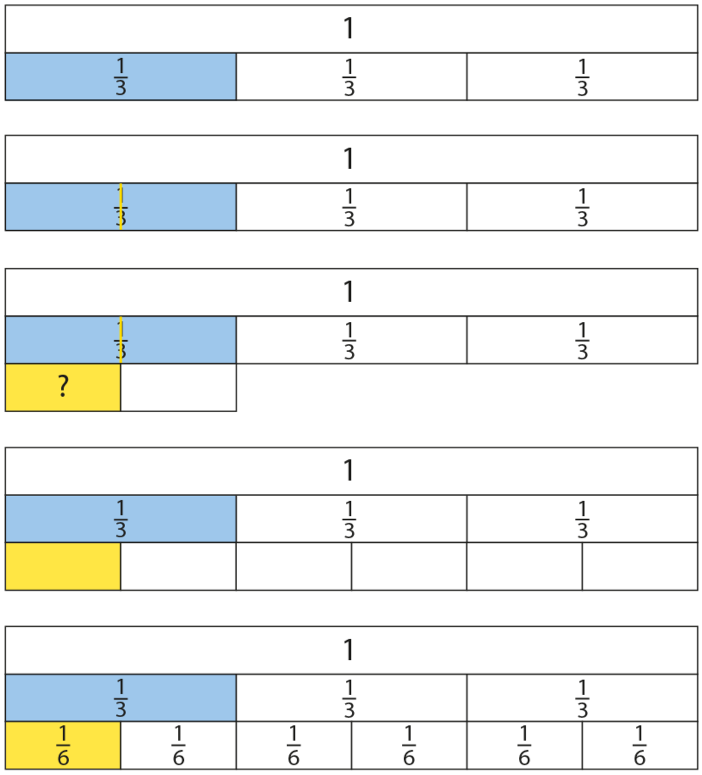
Next, consider half of $\frac{1}{6}$ (which can also be thought of as $\frac{1}{6}$ made half as long).

In equation form, this can be written as $\frac{1}{2}\times\frac{1}{6}=\frac{1}{12}$ and $\frac{1}{6}\times\frac{1}{2}=\frac{1}{12}$.

By now, children should be reasonably familiar with the series of images, so this time start with the third representation from the sequence, as shown opposite. Pose a question such as *'What fraction of the whole is represented by the part with the question mark?'*. There would be 12 equal parts of this size in the whole, so each equal part is $\frac{1}{12}$ of the whole.

Finding half of $\frac{1}{3}$:

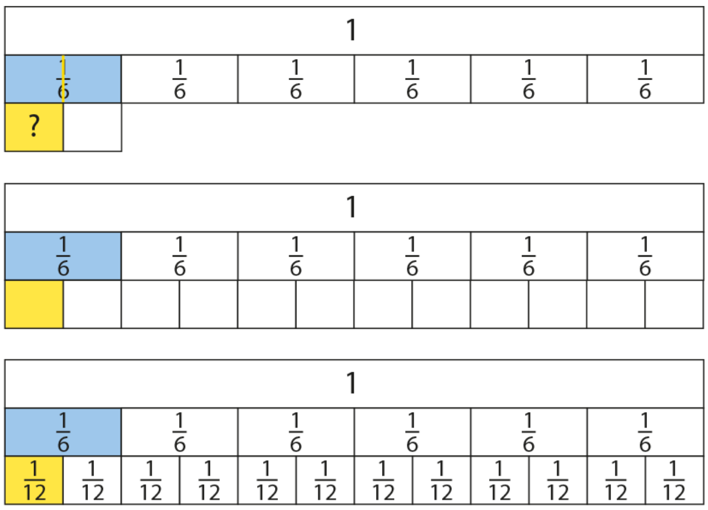
'If one of the thirds is made half as long, what fraction of the whole will one of the new equal parts be?'



- *'There were three equal parts in the whole, each of the three parts was halved, so now we have six equal parts in the whole.'*

Finding half of $\frac{1}{6}$:

'What fraction of the whole is represented by the part with the question mark?'



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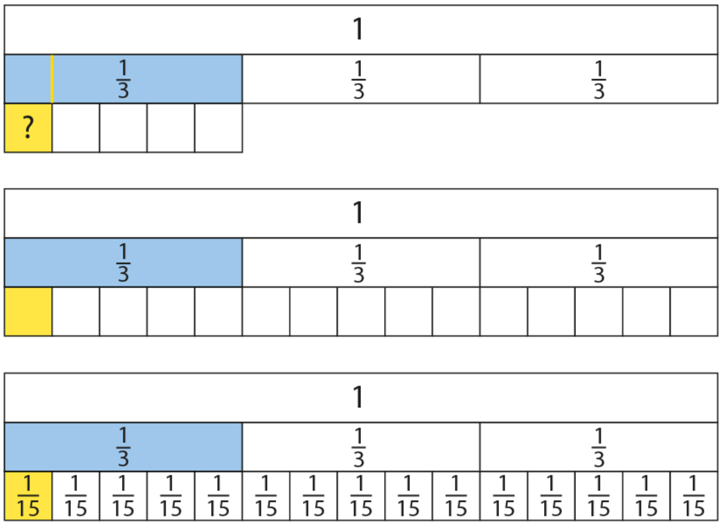
1:3

Now study an example which doesn't involve halving, such as $\frac{1}{5}$ of $\frac{1}{3}$, and work through the same steps. This can be written as both $\frac{1}{5}\times\frac{1}{3}=\frac{1}{15}$ and $\frac{1}{3}\times\frac{1}{5}=\frac{1}{15}$.

While the conceptual language of *'one-fifth of one-third is one-fifteenth'* works well for the first one, the conceptual language for the second, *'one-third, fifthed'*, is not quite as natural. It is important, though, that children have some conceptual understanding of what $\frac{1}{3}\times\frac{1}{5}=\frac{1}{15}$ is expressing, as this will support them in understanding why $\frac{1}{3}\div\mathbf{5}$ can be rewritten as $\frac{1}{3}\times\frac{1}{5}$ when they progress to division of fractions.



1:4

At this point, you might like to display all the pairs of equations considered so far on the board. Ask the children what they notice about the products compared with the fractions being multiplied. Children should notice that the numerators are all one.

They may also notice that the denominator of the product is the product of the other two denominators. Discuss whether this will always be the case when multiplying unit fractions:

- If two equal parts are each split into four equal parts, the result will be eight equal parts, so $\frac{1}{2}\times\frac{1}{4}=\frac{1}{8}$.
- If three equal parts are each split into seven equal parts, it will result in 21 equal parts, so $\frac{1}{3}\times\frac{1}{7}=\frac{1}{21}$.
- If six equal parts are each split into five equal parts, it will result in 30 equal parts, so $\frac{1}{6}\times\frac{1}{5}=\frac{1}{30}$.

Pairs of equations:

$\frac{1}{2}\times\frac{1}{4}=\frac{1}{8}$ $\quad$ $\frac{1}{4}\times\frac{1}{2}=\frac{1}{8}$

$\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$ $\quad$ $\frac{1}{3}\times\frac{1}{2}=\frac{1}{6}$

$\frac{1}{2}\times\frac{1}{6}=\frac{1}{12}$ $\quad$ $\frac{1}{6}\times\frac{1}{2}=\frac{1}{12}$

$\frac{1}{3}\times\frac{1}{5}=\frac{1}{15}$ $\quad$ $\frac{1}{5}\times\frac{1}{3}=\frac{1}{15}$

Multiplying by a unit fraction makes a number smaller:

Equation	Comparison
$\frac{1}{2}\times\frac{1}{4}=\frac{1}{8}$	$\frac{1}{8}<\frac{1}{2}$ and $\frac{1}{8}<\frac{1}{4}$
$\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$	$\frac{1}{6}<\frac{1}{2}$ and $\frac{1}{6}<\frac{1}{3}$
$\frac{1}{2}\times\frac{1}{6}=\frac{1}{12}$	$\frac{1}{12}<\frac{1}{2}$ and $\frac{1}{12}<\frac{1}{6}$
$\frac{1}{5}\times\frac{1}{3}=\frac{1}{15}$	$\frac{1}{15}<\frac{1}{3}$ and $\frac{1}{15}<\frac{1}{5}$

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	It is always the case that unit fractions can be multiplied by multiplying the denominators. Now might be a good point to introduce the generalisation: ***'When multiplying unit fractions, multiply the denominators.'*** If it hasn't already arisen in discussion, ask the children whether the products in these multiplications are larger or smaller than the fractions being multiplied. By now, the children should be extremely confident in recognising that, when comparing unit fractions, the bigger the denominator, the smaller the fraction. As such, they should recognise that (in the examples they have met so far) the product is always smaller than the numbers being multiplied. Refer back to the bar models in steps *1:1* and *1:2* to provide a visual reinforcement of this. Multiplying a number by a unit fraction means that we are finding a unit fraction of that number. This is *always* going to be less than the whole, so multiplying by a unit fraction makes a number (including a fractional number) smaller. You could summarise this with the following generalisation: ***'When multiplying unit fractions, the product is smaller than the fractions being multiplied.'***	
1:5	Next, you could display images that relate back to the examples met so far, and show multiplying by other unit fractions. Present an example, such as the one on the next page, and ask the children what equal part is represented by the question mark ($\frac{1}{3}$ of $\frac{1}{2}$). Use the sequence of diagrams to show that if $\frac{1}{2}$ is made $\frac{1}{3}$ of its size, then there will be six equal parts making up the whole. It	

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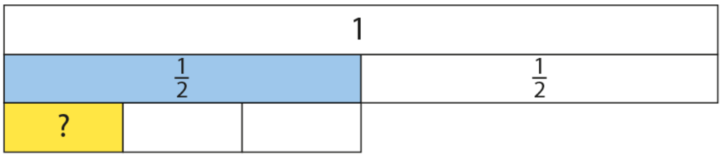
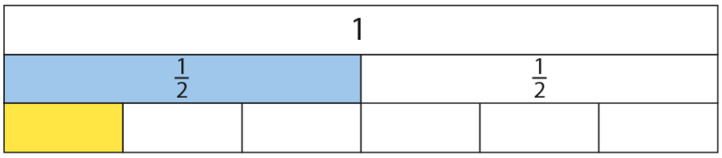
will therefore be $\frac{1}{6}$ of the original whole.

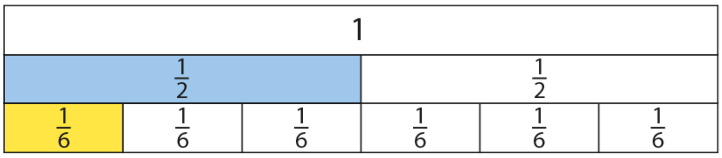
Ask the children to write two different multiplication equations to represent this. Once again, check that the product is smaller than the two fractions being multiplied.

Compare these equations with the image and equations children wrote for the example in step *1:2*. They will notice that the pair of equations is exactly the same. Each of these equations can represent both $\frac{1}{3}$ of $\frac{1}{2}$ and $\frac{1}{2}$ of $\frac{1}{3}$ but either way the resulting fraction is $\frac{1}{6}$.

'What equal part is represented by the question mark?'







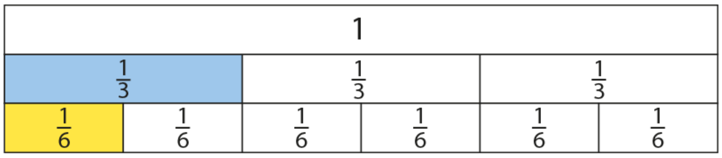
$\frac{1}{3}$ of $\frac{1}{2}$

$$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$

and

$$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

Comparing with equations in step *1:2*:



$\frac{1}{2}$ of $\frac{1}{3}$

$$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

and

$$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$

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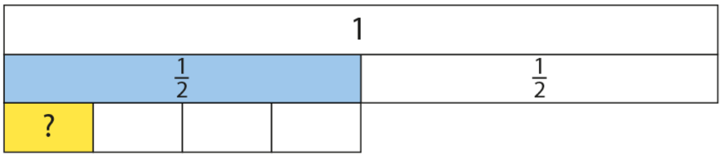
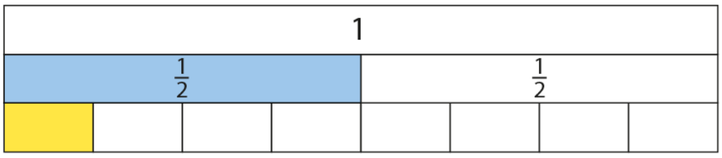
1:6

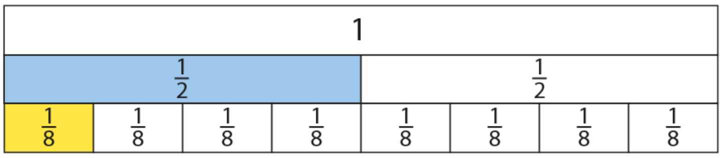
Now consider $\frac{1}{4}$ of $\frac{1}{2}$. You may choose to ask the children to draw a similar image to the ones provided opposite, and then compare this with the images of $\frac{1}{2}$ of $\frac{1}{4}$.

As before, exactly the same pair of equations can be used to represent both structures. In both structures the resulting fraction is $\frac{1}{8}$.

Just as multiplication with whole numbers is commutative, multiplication with proper fractions is also commutative.





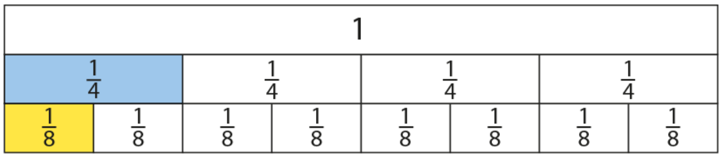


$\frac{1}{4}$ of $\frac{1}{2} = \frac{1}{8}$

$\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$

and

$\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$



$\frac{1}{2}$ of $\frac{1}{4} = \frac{1}{8}$

$\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$

and

$\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$

1:7

At this point, provide children with varied practice, including multiplying unit fractions. Allow them ample opportunity to practise, checking their understanding so far.

You may wish to remind children that they should always ask themselves whether the product is smaller than the

Completing equations:

'Complete these equations.'

$\frac{1}{3} \times \frac{1}{4} =$ $\frac{1}{5} \times \frac{1}{4} =$ $\frac{1}{6} \times \frac{1}{3} =$

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	two fractions being multiplied. Include dòng nǎo jīn problems, such as those given opposite, to consolidate and deepen understanding.	True-false style problems: *'True or false?'*

	True (✓) or false (×)?
$\frac{1}{8} \times \frac{1}{3} = \frac{2}{24}$	
$\frac{1}{5} \times \frac{1}{6} = \frac{1}{15}$	
$\frac{1}{6} \times \frac{1}{4} = \frac{1}{4} \times \frac{1}{6}$	

Dòng nǎo jīn:

- *'Which single digits could go in the boxes to make these statements true?'*

$$\frac{1}{\square} \times \frac{1}{\square} = \frac{1}{18}$$

$$\frac{1}{10} > \frac{1}{\square} \times \frac{1}{\square} > \frac{1}{18}$$

- *'Fill in the missing symbols (<, > or =).'*

$$\frac{1}{2} \times 6 \bigcirc \frac{1}{2}$$

$$\frac{1}{2} \times 1 \bigcirc \frac{1}{2}$$

$$\frac{1}{2} \times \frac{1}{6} \bigcirc \frac{1}{2}$$

$$\frac{1}{2} \times \frac{1}{6} \bigcirc \frac{1}{6}$$

- *'Write a different number in each statement to make it true.'*

$$\frac{1}{7} < \frac{1}{7} \times \square \qquad \frac{1}{7} = \frac{1}{7} \times \square$$

$$\frac{1}{7} > \frac{1}{7} \times \square$$

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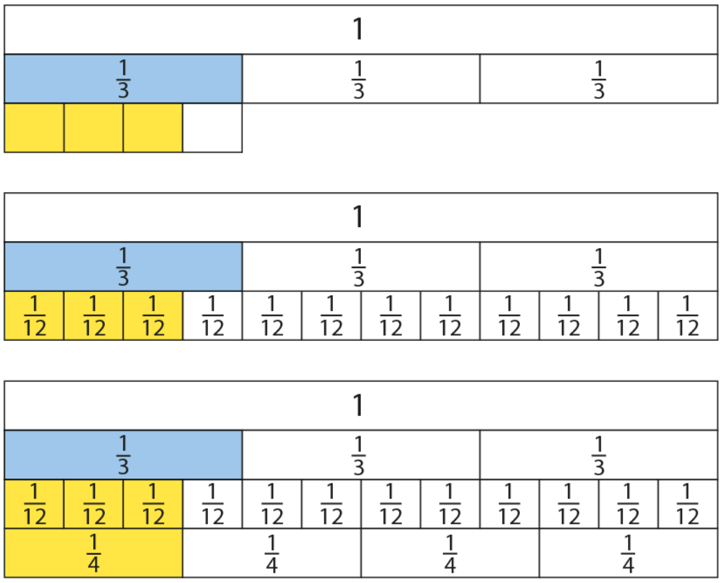
1:8	Repeat the procedure but with problems set in a real-life context. For example: *'Rudi drank $\frac{3}{4}$ of a $\frac{1}{3}$ litre bottle of juice. What fraction of a litre did Rudi drink?'* Discuss with children, reaching the conclusion that this calculation can be expressed either as $\frac{3}{4}\times\frac{1}{3}$ or $\frac{1}{3}\times\frac{3}{4}$. Examine the model provided opposite: • First the $\frac{1}{3}$ is split into four equal parts. • Three parts are shaded. • There would be twelve of these equal parts making up one whole, so each equal part is $\frac{1}{12}$. • Three parts are shaded, so the shaded section is $\frac{3}{12}$. • $\frac{3}{12}$ is equivalent to $\frac{1}{4}$, so Rudi drank $\frac{1}{4}$ of a litre of juice. • $\frac{3}{4}\times\frac{1}{3}=\frac{3}{12}=\frac{1}{4}$. Check children's understanding by asking questions such as: *'Is $\frac{1}{4}$ smaller than the fractions we started with?'*. Through reasoning it is evident that $\frac{1}{4}$ is smaller than both $\frac{3}{4}$ and $\frac{1}{3}$.	*'Rudi drank $\frac{3}{4}$ of a $\frac{1}{3}$ litre bottle of juice. What fraction of a litre did Rudi drink?'*  $\frac{3}{4}\times\frac{1}{3}=\frac{3}{12}=\frac{1}{4}$
1:9	At this stage, progress to multiplying two non-unit fractions. You may wish to present an example of multiplying pairs of non-unit fractions before extending the generalisation to unit fractions. Look at $\frac{4}{5}$ of $\frac{2}{3}$ (Model 1 on the next page), which can be written as $\frac{4}{5}\times\frac{2}{3}$ or $\frac{2}{3}\times\frac{4}{5}$.	

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- First consider the whole. If the whole is divided into three equal parts, each part is $\frac{1}{3}$. $\frac{2}{3}$ are needed, so shade two of the thirds.
- Now it is necessary to think about the fifths of the thirds. Draw another bar to show each third divided into five equal parts. Each part represents $\frac{1}{5}$ of a third and (because there are fifteen in total) $\frac{1}{15}$ of the whole.
- $\frac{4}{5}$ of $\frac{2}{3}$ are needed, so shade four parts of two of the thirds.
- Ask: *'How many parts are shaded in total?'*.
  There are eight parts shaded, and each part is $\frac{1}{15}$ of the whole, so $\frac{4}{5}$ of $\frac{2}{3}$ is $\frac{8}{15}$.

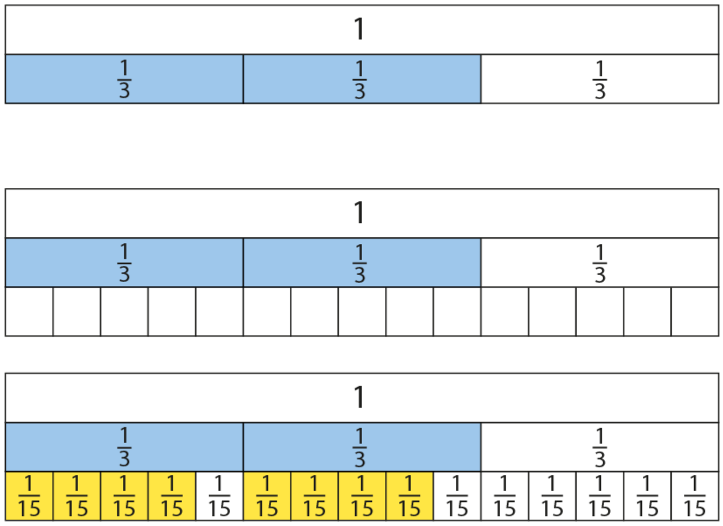
The model demonstrates that $\frac{4}{5}$ of $\frac{1}{3}$ is equal to $\frac{4}{15}$. So, $\frac{4}{5}$ of $\frac{2}{3}$ is equal to $\frac{8}{15}$.

Show the two completed equations alongside the model.

This multiplication is also shown opposite in an alternative model, as discussed in the *Overview of learning*. To use this model, you may find it useful to follow these steps:

- Begin with a rectangle split into thirds horizontally and shade two of them.
- Next, split the rectangle into fifths vertically. There are now fifteen equal parts.
- $\frac{4}{5}$ of the sections that were shaded blue can now be shaded yellow to show the solution.
- The area shaded yellow is $\frac{8}{15}$ of the whole rectangle.

Model 1:

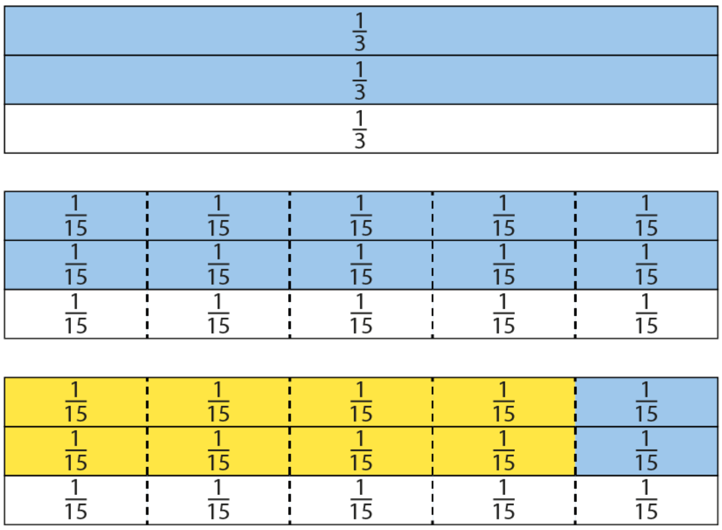


$$\frac{4}{5} \times \frac{2}{3} = \frac{8}{15}$$

and

$$\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$$

Model 2:



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1:10

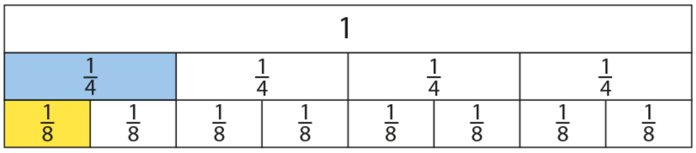
At this point, display the three calculations the children have just examined, without the simplified forms, and draw their attention to the numerators and denominators. Refer back to the representations, if needed. Ask children what they notice, trying to extract responses along the lines of:

- *'The numerator of the product is the product of the numerators being multiplied.'*
- *'The denominator of the product is the product of the denominators being multiplied.'*

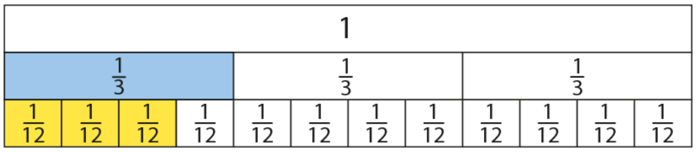
Confirm for children that this is the case for any pair of fractions being multiplied. The number of parts can be found by multiplying the denominators. The number of shaded parts can be found by multiplying the numerators. In $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$, for example, the total number of parts in the whole is three groups of five (multiplying the denominators), and the total number of shaded parts is two groups of four (multiplying the numerators).

You may now choose to introduce the following generalisation: ***'To multiply fractions, we can multiply the numerators and multiply the denominators.'***

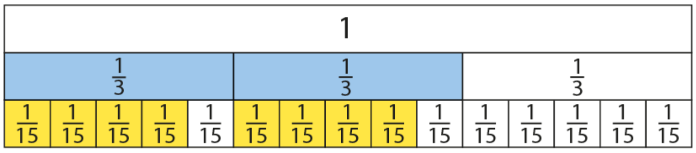
Note that this generalisation is satisfactory for use in primary schools, but is not necessarily the most efficient method. In secondary school, children will be taught to check whether fractions can be simplified *before* multiplying by cancelling common factors in the numerators and denominators. This is not expected for primary-school children, but can be a useful approach, particularly when working with larger denominators. At primary level, it is acceptable to



$\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$ $\qquad$ $\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$



$\frac{3}{4} \times \frac{1}{3} = \frac{3}{12}$ $\qquad$ $\frac{1}{3} \times \frac{3}{4} = \frac{3}{12}$



$\frac{4}{5} \times \frac{2}{3} = \frac{8}{15}$ $\qquad$ $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$

Simplifying products:

$$\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$$ (can't be simplified)

$$\frac{3}{4} \times \frac{1}{3} = \frac{\cancel{3}^{1}}{\cancel{12}_{4}} = \frac{1}{4}$$

$$\frac{4}{5} \times \frac{2}{3} = \frac{8}{15}$$ (can't be simplified)

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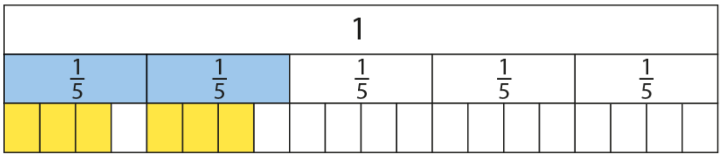
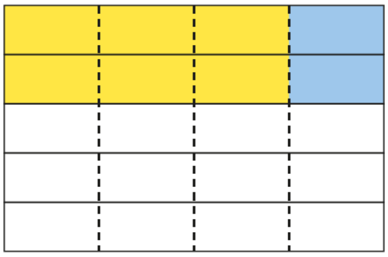
	check if the product can be simplified after the multiplication has been performed, as the children learnt in segment *3.7 Finding equivalent fractions and simplifying fractions*. In the case of the three examples here, the product can be simplified in the second example, but not in the other two examples.	
1:11	Work through guided practice as a class. You could use the examples provided opposite, alongside your own. (Note: for the modelled example, select the format of model that you have been using in class – not both.) As you progress, try to move away from using a model and just apply the procedure, simplifying if necessary. Each time, sense-check whether the product is bigger or smaller than the numbers that were multiplied. It is important that children fully appreciate that when they multiply by a proper fraction, they are making a number smaller. Repeat this statement after calculating each example.	• *'Complete these calculations.'* $\frac{3}{4}\times\frac{1}{2}=$ $\frac{1}{3}\times\frac{2}{3}=$ $\frac{2}{3}$ of $\frac{1}{5}$ $\frac{5}{7}\times\frac{3}{8}=$ $\frac{3}{7}\times\frac{5}{8}=$ $\frac{7}{8}\times\frac{3}{5}=$ • *'Look at this model. What calculation does it represent?'*  
1:12	Provide children with examples that can be calculated using the generalisation: ***'To multiply fractions, we can multiply the numerators and multiply the denominators.'*** Children should be reminded to check whether the products can be simplified, and to check that the product is smaller than the fractions being multiplied. You can explore this concept further using dòng nǎo jīn problems like the ones shown. Allow children to discuss their ideas and compare answers.	$\frac{4}{5}\times\frac{1}{3}=$ $\frac{1}{7}\times\frac{5}{8}=$ $\frac{2}{9}\times\frac{7}{8}=$ $\frac{3}{4}\times\frac{5}{6}=$ $\frac{4}{5}\times\frac{5}{12}=$ $\frac{3}{10}\times\frac{5}{6}=$

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		Dòng nǎo jīn: • *'Fill in the missing numbers.'* $\frac{1}{2} \times \frac{\square}{\square} = \frac{5}{16}$ $\quad\quad$ $\frac{3}{4} \times \frac{\square}{\square} = \frac{3}{20}$ • *'Solve these two calculations.'* $\frac{5}{7} \times \frac{3}{8} =$ $\quad\quad$ $\frac{3}{7} \times \frac{5}{8} =$ • *'What do you notice?'* • *'Why is this the case?'* • *'How many solutions can you find to make the statement true?'* $\frac{\square}{\square} \times \frac{\square}{\square} \times \frac{\square}{\square} = \frac{3}{20}$ • *'Solve these calculations using reasoning.'* $\frac{7}{7} \times \frac{1}{3} =$ $\quad\quad$ $\frac{2}{9} \times \frac{3}{3} =$ • *'What do you notice in these calculations?'* $\frac{3}{4} \times \frac{\square}{\square} = \frac{6}{8}$ $\quad\quad$ $\frac{4}{4} \times \frac{\square}{\square} = \frac{3}{8}$

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Teaching point 2:

When a fraction is divided by a whole number, it makes it smaller. To divide a fraction by a whole number, convert it to an equivalent multiplication.

Steps in learning

2:1

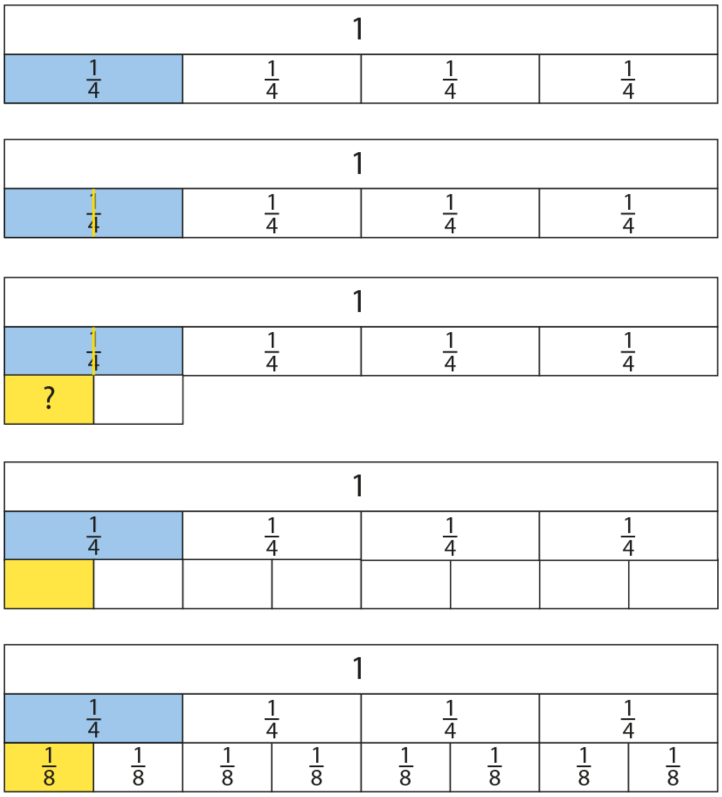
Guidance

Return to the image (shown opposite) from step *1:1*. Up to this point, children have learnt that two different expressions can represent halving of $\frac{1}{4}$: $\frac{1}{2} \times \frac{1}{4}$ and $\frac{1}{4} \times \frac{1}{2}$. However, there is another expression that can be used to represent this: $\frac{1}{4} \div 2$.

Discuss how this expression relates to the image: $\frac{1}{4}$ of the whole has been shaded blue. This quarter is then divided into two equal parts, represented by the yellow section. Ask children what fraction the yellow section is of the whole. To do this, they need to identify how many of the yellow sections will fit into one whole.

There are eight equal parts in the whole, so each equal part is $\frac{1}{8}$. The yellow section is $\frac{1}{8}$ of the whole, so $\frac{1}{4} \div 2 = \frac{1}{8}$.

Representations



2:2

Guidance

Look at the following equations alongside each other:

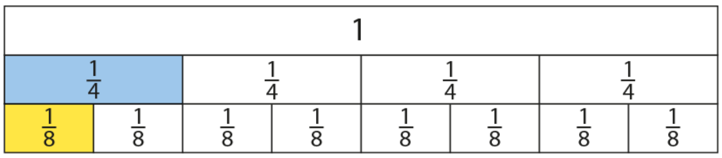
- $\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$
- $\frac{1}{4} \div 2 = \frac{1}{8}$

Ask:

- *'What is the same?'*
- *'What is different?'*

Refer back to the final image in the sequence in step *2:1*. Both of these equations express exactly the same

Representations



$\frac{1}{4} \times \frac{1}{2} = \frac{1}{4} \div 2 =$

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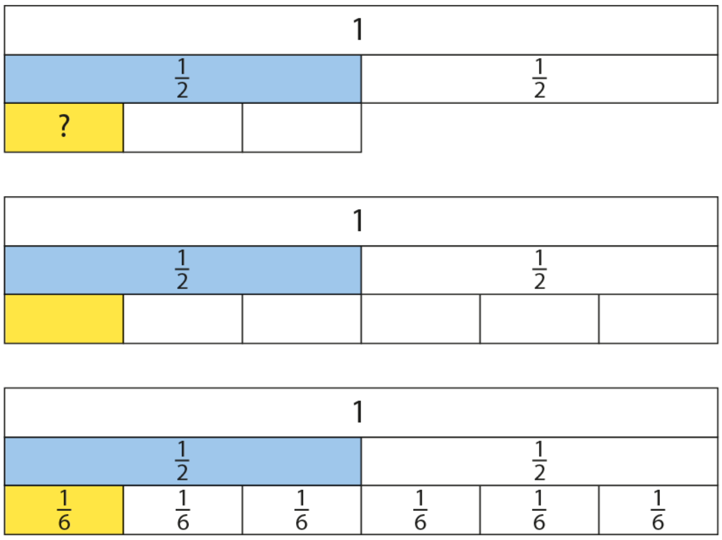
	thing: a halving, or dividing by two, of the number $\frac{1}{4}$. In segment *3.6 Multiplying whole numbers and fractions*, children learnt that multiplying a whole number by $\frac{1}{2}$ is equivalent to dividing that whole number by two and, as shown here, this extends to dividing a fraction by two. $\frac{1}{4}\times\frac{1}{2}=\frac{1}{4}\div\mathbf{2}$ (Note that we now move to writing the multiplication expression with the multiplicand ($\frac{1}{4}$) first. This is in order to make the link with $\frac{1}{4}\div\mathbf{2}$ as clear as possible. This can be read 'conceptually' as $\frac{1}{4}$ 'halved' is equivalent to $\frac{1}{4}$ divided by two.)	
2:3	Return to another example from *Teaching point 1* (step *1:2*). The following two expressions have already been learnt: $\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$ and $\frac{1}{3}\times\frac{1}{2}=\frac{1}{6}$ As in the previous example (step *2:2*), this can also be represented as a division expression. In the model, $\frac{1}{2}$ of the whole bar has been shaded blue. That half is being divided by three, represented by the yellow section. The division expression is $\frac{1}{2}\div\mathbf{3}$. Again, ask children to identify what fraction the yellow section is of the whole. They then need to find how many of these yellow sections will fit into one whole. There are six equal parts of that size altogether in the whole, so the yellow section is $\frac{1}{6}$ of the whole. This can be	 $\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$ $\frac{1}{2}\div\mathbf{3}=\frac{1}{6}$

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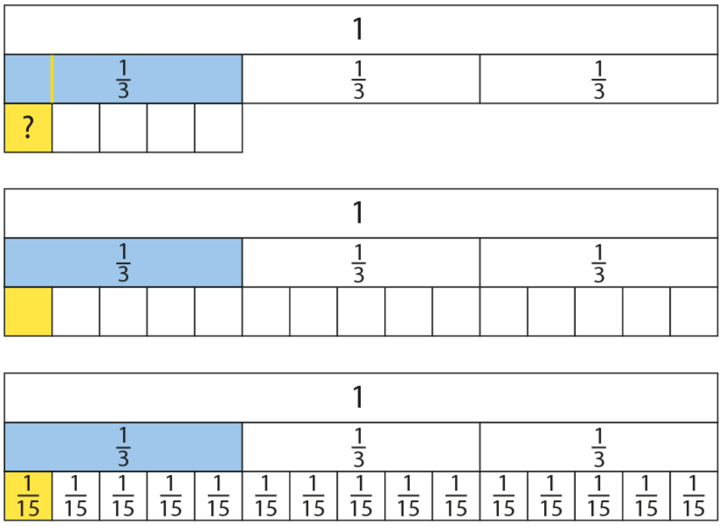
	expressed as $\frac{1}{2} \div 3 = \frac{1}{6}$. As before, look at the following pair of equations alongside each other and relate them to the final diagram. • $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$ • $\frac{1}{2} \div 3 = \frac{1}{6}$ Ask: • *'What is the same?'* • *'What is different?'* Both equations represent exactly the same thing; a 'thirding', or division by three, of the number $\frac{1}{2}$. It is therefore possible to say that $\frac{1}{2} \times \frac{1}{3} = \frac{1}{2} \div 3$.	
2:4	Repeat with a third image from *Teaching point 1* (step *1:3*). As previously discussed, this image can be represented by the expression $\frac{1}{3} \times \frac{1}{5}$. Ask: *'What division expression can I use to represent this?'* Provide the scaffold of this partially-completed expression for the children. $\frac{1}{3} \div \square$ In the first image opposite, $\frac{1}{3}$ of the whole bar has been shaded blue. $\frac{1}{3}$ is being divided into five equal parts, represented by the yellow section. The completed expression is $\frac{1}{3} \div 5$. There are fifteen equal parts of that size altogether in the whole, so the yellow section is $\frac{1}{15}$ of the whole. Therefore, $\frac{1}{3} \div 5 = \frac{1}{15}$. Look at the multiplication and	 $\frac{1}{3} \div \square$

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division equations alongside each other. As previously, discuss the connections between them.

- $\frac{1}{3}\times\frac{1}{5}=\frac{1}{15}$
- $\frac{1}{3}\div 5=\frac{1}{15}$

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2:5

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Display all the pairs of equations already seen in this teaching point, and summarise as follows:

- Dividing by two is the same as multiplying by $\frac{1}{2}$.
- Dividing by three is the same as multiplying by $\frac{1}{3}$.
- Dividing by five is the same as multiplying by $\frac{1}{5}$.

Relate each of these statements back to the models, so that children can understand why each one is true. Confirm that this can be extended to division by any integer. To divide any fraction by a whole number, we can convert it to an equivalent multiplication.

When you feel it is appropriate, introduce the generalisation/stem sentence: ***'To divide a fraction by a whole number, we can change it to an equivalent multiplication. To divide by ___, we can multiply by ___.'***

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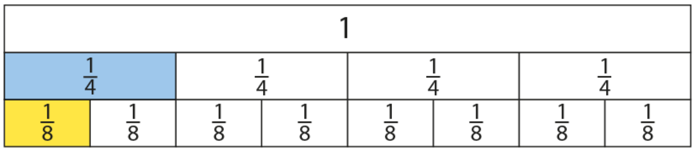
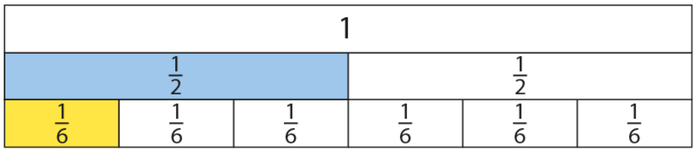
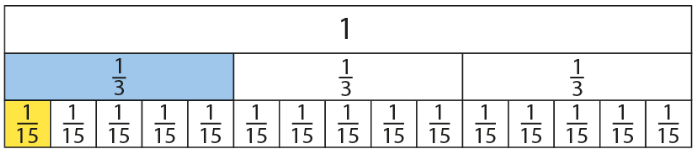


$\frac{1}{4}\div 2=\frac{1}{8}\longrightarrow\frac{1}{4}\times\frac{1}{2}=\frac{1}{8}$



$\frac{1}{2}\div 3=\frac{1}{6}\longrightarrow\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$



$\frac{1}{3}\div 5=\frac{1}{15}\longrightarrow\frac{1}{3}\times\frac{1}{5}=\frac{1}{15}$

'To divide a fraction by a whole number, we can change it to an equivalent multiplication. To divide by five, we can multiply by $\frac{1}{5}$.'

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2:6

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Now present the division expression opposite, and ask the children to use what they have learnt so far to convert the division to an equivalent multiplication.

Initially avoid presenting a supporting image. After discussion of the equivalent calculations, use the image opposite as a reference point to tie the two expressions together.

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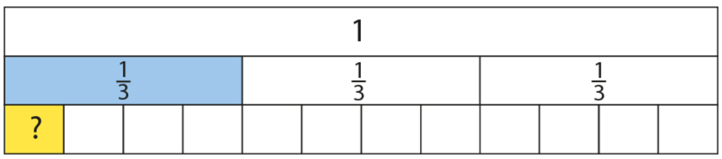
$\frac{1}{3}\div 4$

$\frac{1}{3}\times\frac{\square}{\square}$



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2:7

In *Teaching point 1* children learnt how to multiply pairs of fractions, and they should be confident with the simple procedure for this. This procedure (i.e. multiplying the numerators and multiplying the denominators) can be used to calculate the answer to $\frac{1}{3} \times \frac{1}{4}$.

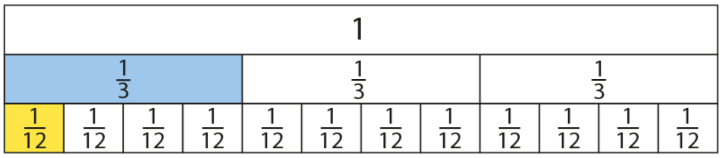
Complete the calculation using this approach to calculate an answer of $\frac{1}{12}$.

Look at the completed calculation opposite. The equivalent multiplication is used to calculate an answer to the initial division.

Once you have completed the calculation as a class, refer back to the image (now labelled in twelfths) to check the answer. Ask the children to explain both equations, referencing the image in their explanation.

$$\frac{1}{3} \div 4 = \frac{1}{12}$$

$$\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$$



2:8

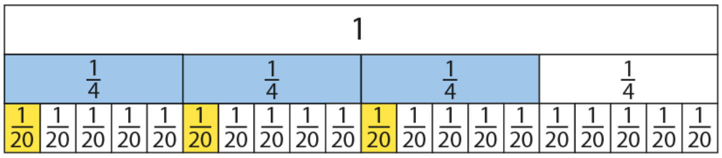
As a final step before providing practice opportunities, look at how this approach to dividing fractions by whole numbers can be extended to non-unit fractions.

As in the previous step, present a calculation (a non-unit fraction divided by a whole number) and ask the children to convert to an equivalent multiplication, e.g.:

- $\frac{3}{4} \div 5$
- $\frac{3}{4} \times \frac{1}{5}$

Completing the multiplication using the approach learnt in *Teaching point 1* gives an answer of $\frac{3}{20}$. Refer to the image opposite to demonstrate that this is indeed the correct answer.

The children know by now that dividing by five and multiplying by $\frac{1}{5}$ are equivalent calculations, and this knowledge can be applied to whole



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<table>
<tr><td></td><td>numbers, unit fractions and non-unit fractions. Any division of a fraction by a whole number can be rewritten as a multiplication of a fraction by a fraction. This is a really useful approach for solving calculations involving division of a fraction by a whole number, as children are already confident with the multiplication of pairs of fractions. However, try to move beyond this conversion being something they 'do' to get an answer to a division calculation, and emphasise that the concept of what they are doing is also important.

Understanding that division is equivalent to multiplication by a fraction helps children understand that multiplication can make numbers smaller. As mentioned in the overview, in some branches of higher mathematics, division ceases to be a concept that is used, since any division can be replaced by a multiplication (for example, dividing by four becomes multiplying by one-quarter).</td><td></td></tr>
<tr><td>2:9</td><td>Provide practice in dividing a fraction by a whole number. Begin by guiding children through examples as a class, and then provide them with independent practice. You might want to offer varied practice, including:
• completing equivalent calculations
• solving division calculations by first converting to a multiplication; use some with resulting fractions that can be simplified and some that can't
• real-life contexts.

To further explore the concepts discussed in this teaching point, present dòng nǎo jīn problems such as the examples provided. These missing-number problems will challenge children to consider all the different elements of the calculation.</td><td>Equivalent calculations:
• 'Fill in the missing numbers.'
$\frac{1}{8} \div 3 = \frac{1}{8} \times \square$ $\frac{1}{4} \div 4 = \square \times \square$
$\frac{5}{9} \div 2 = \frac{5}{9} \times \square$ $\frac{2}{3} \div 7 = \square \times \square$
$\frac{1}{7} \times \frac{1}{4} = \frac{1}{7} \div \square$ $\frac{3}{5} \times \frac{1}{2} = \frac{3}{5} \div \square$</td></tr>
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- *'Circle the calculations that are equivalent to* $\frac{3}{5} \times \frac{1}{5}$.*'*

$\frac{1}{5} \times \frac{3}{5}$ $\qquad$ $\frac{3}{5} \div 5$

3×1 $\qquad$ $\frac{3}{5} \times 5$

Solving division calculations:

'Solve these division calculations.'

$\frac{1}{3} \div 7 =$ $\qquad$ $\frac{5}{8} \div 4 =$ $\qquad$ $\frac{4}{9} \div 3 =$

$\frac{3}{8} \div 6 =$ $\qquad$ $\frac{4}{5} \div 8 =$

Real-life contexts:

- *'Sofia is running a half-marathon. She is* $\frac{1}{3}$ *of the way through the race. What fraction of a full marathon has she completed?'*
- *'Joel has* $\frac{3}{4}$ *litre of juice. He divides it equally between 5 glasses. What fraction of a litre is in each glass?'*

Dòng nǎo jīn:

'Fill in the missing numbers to make these statements true.'

$\frac{2}{9} \div \square = \frac{2}{63}$ $\qquad$ $\frac{\square}{\square} \div 5 = \frac{3}{35}$

$\frac{\square}{10} \div 3 = \frac{\square}{6}$ $\qquad$ $\frac{4}{5} \div \square = \frac{1}{10}$

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Teaching point 3:

A more efficient method can be used to divide a fraction by a whole number when the whole number is a factor of the numerator.

Steps in learning

<table>
<tr><th></th><th>Guidance</th><th>Representations</th></tr>
<tr><td>3:1</td><td>
Look again at the generalisation/stem sentence from step 2:5: 'To divide a fraction by a whole number, we can change it to an equivalent multiplication. To divide by ___, we can multiply by ___.' It works for all divisions of a fraction by a whole number, but is not always the most efficient method. In some cases, there is an alternative way to think about it. This relies on unitising rather than scaling by conversion to an equivalent multiplication.
Unitising applies the idea that if, for instance, I have six one-eighths and I am dividing them into two equal groups, there will be three one-eighths in each group. This is an application of the division fact $6 \div 2 = 3$, which has been generalised to six units divided by two is equal to three units. We can derive (for example) that:

6 tens ÷ 2 = 3 tens
6 cm ÷ 2 = 3 cm
$\frac{6}{8} \div 2 = \frac{3}{8}$ (in this case).

In order for this approach to work, the numerator has to be a multiple of the divisor. $\frac{6}{8} \div \mathbf{2}$ can be calculated using this method because the six (one-eighths) can be divided by two. If the calculation was $\frac{5}{8} \div \mathbf{2}$, this approach wouldn't work, and the equivalent multiplication method $\frac{5}{8} \times \frac{1}{2}$ would be needed.
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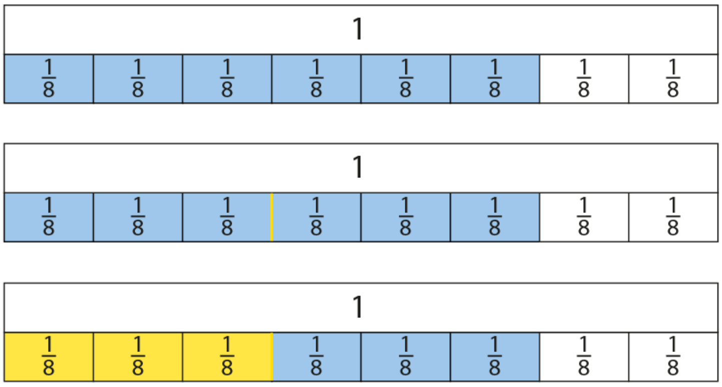

'$\frac{6}{8}$ is six one-eighths. If we divide six one-eighths into two equal groups then each of the groups has three one-eighths or $\frac{3}{8}$ in it.'
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<table>
<tr>
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<td>Show the children the images on the previous page of $\frac{6}{8}$ being divided into two equal groups. Each of those equal groups contains $\frac{3}{8}$. Use the familiar unitising language to discuss what is happening.</td>
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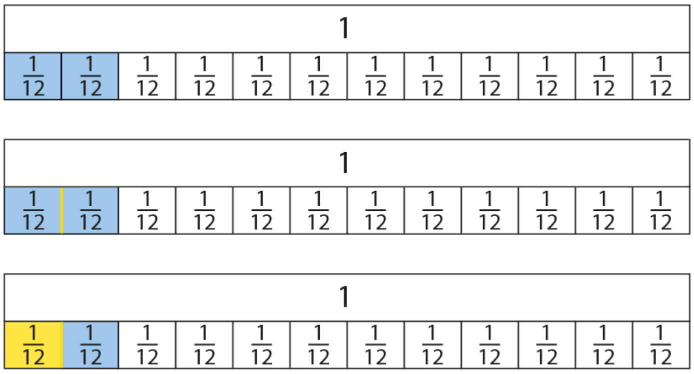
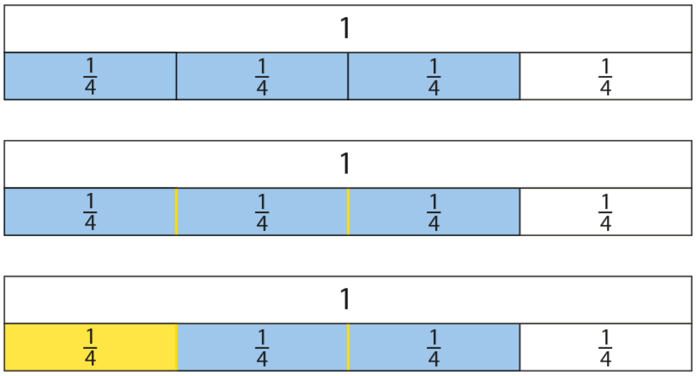
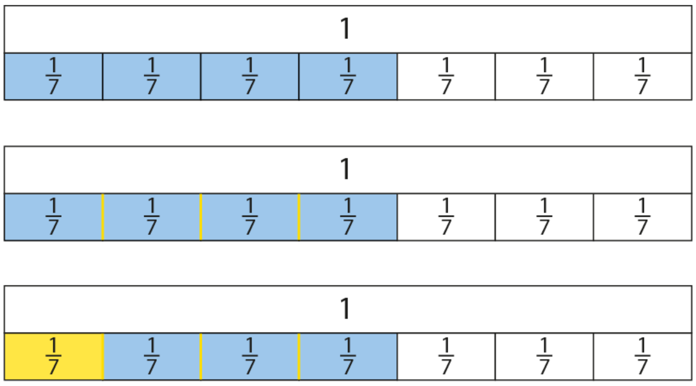
<tr>
<td>3:2</td>
<td>Display a sequence of division calculations and images to which this unitising method can be successfully applied. See the examples opposite. These all have the same numeral for the numerator and divisor.

Discuss the calculations, using the unitising language as in step 3:1, and generalise that when the divisor is the same numeral as the numerator, the resultant quotient will be a unit fraction.</td>
<td>$\frac{2}{12} \div 2 = \frac{1}{12}$


$\frac{3}{4} \div 3 = \frac{1}{4}$


$\frac{4}{7} \div 4 = \frac{1}{7}$

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</table>

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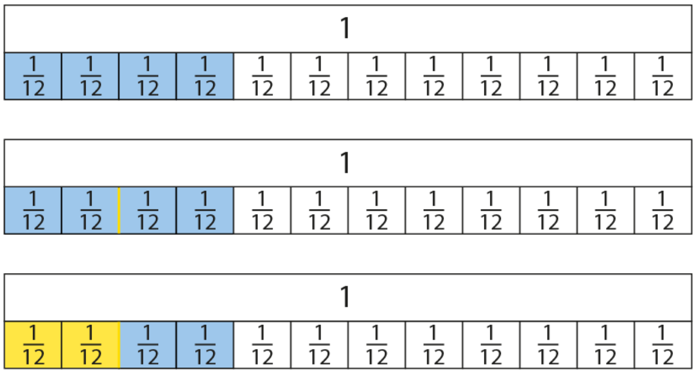
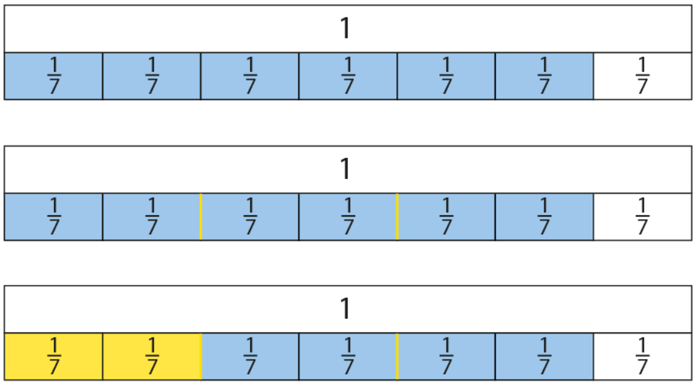
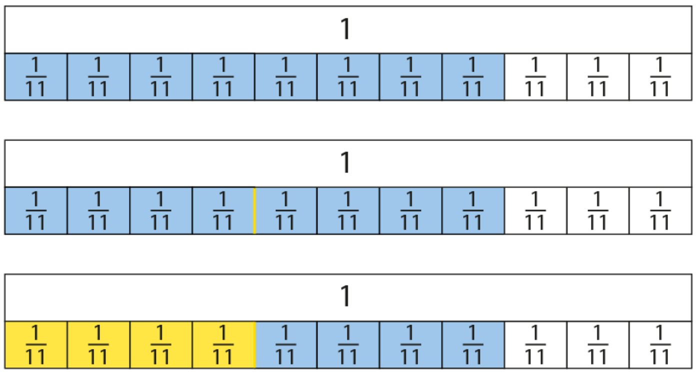
3:3	Now look at a non-unit fraction (e.g. $\frac{4}{12} \div \mathbf{2}$). The image opposite will support children in seeing that $\frac{4}{12} \div \mathbf{2} = \frac{2}{12}$. Again, summarise what is happening using the unitising language: '$\frac{4}{12}$ *is four one-twelfths. If we divide four one-twelfths into two equal groups, then each of the groups is two one-twelfths, because four divided by two is two.*' Repeat for another calculation, such as $\frac{6}{7} \div \mathbf{3}$, still with an accompanying representation for support, and using the unitising language: '$\frac{6}{7}$ *is six one-sevenths. If we divide six one-sevenths into three equal groups, then each of the groups is two one-sevenths because six divided by three is two.*'	$\frac{4}{12} \div \mathbf{2}$  $\frac{6}{7} \div \mathbf{3}$ 
3:4	Present children with further examples of divisions of a fraction by a whole number to explore. Ask them to discuss each example with a partner and justify their thinking (and ultimately their solution), using the unitising language. Some examples are given opposite and on the next page. Only reveal the supporting images after children have first discussed and solved the calculations. Ask the children to explain their answers with reference to the diagrams.	$\frac{8}{11} \div \mathbf{2}$ 

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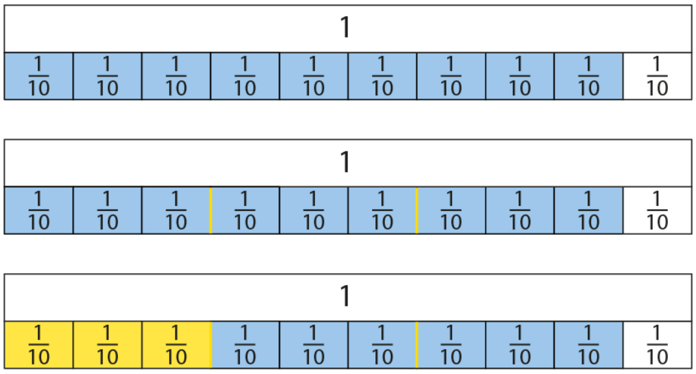
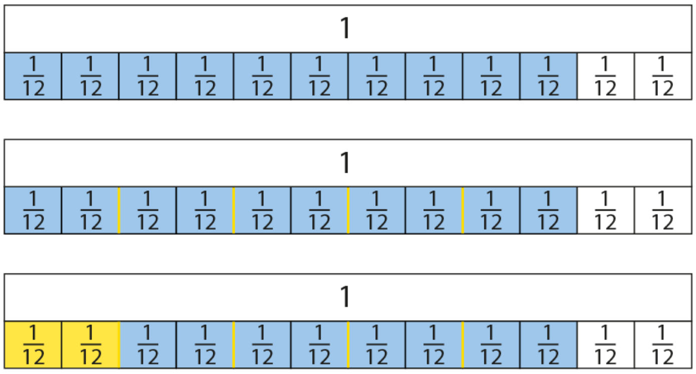
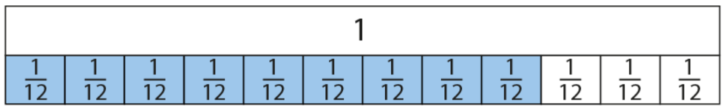
		$\frac{9}{10} \div 3$  $\frac{10}{12} \div 5$ 
3:5	Look at the calculation $\frac{10}{12} \div \mathbf{5}$ and supporting representation that the children have just completed. Adjust the image to show $\frac{9}{12} \div \mathbf{5}$. Examine with the children how it is no longer possible to solve this calculation using this approach: nine is not a multiple of five, so it is not possible to divide nine one-twelfths into groups of whole one-twelfths. In order for this approach to work, the numerator has to be a multiple of the divisor. Summarise that: • if the numerator is a multiple of the divisor, it can be solved with the unitising method the children just learnt • if the numerator is *not* a multiple of the divisor, the calculation must be solved by conversion to an equivalent multiplication.	

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3:6

Now this has been summarised, present children with the following table of calculations. As a class, fill in the second column, identifying the calculations in which the numerator is a multiple of the divisor.

Once you have completed the first two columns, refer back to the two statements in step *3:5*, using the unitising method to solve the calculations where appropriate and the equivalent multiplication method to solve all of them. Discuss the patterns within the completed table as a class. Note that where both methods can be used, the two fractions obtained are not the same, although, of course, they are equivalent in value.

	Is the numerator a multiple of the divisor?	**Unitising method**	**Convert expression to equivalent multiplication**
$\frac{1}{8} \div 2$	No	✘	$\frac{1}{16}$
$\frac{2}{8} \div 2$	Yes	$\frac{1}{8}$	$\frac{2}{16}$
$\frac{3}{8} \div 2$	No	✘	$\frac{3}{16}$
$\frac{4}{8} \div 2$	Yes	$\frac{2}{8}$	$\frac{4}{16}$
$\frac{5}{8} \div 2$	No	✘	$\frac{5}{16}$
$\frac{6}{8} \div 2$	Yes	$\frac{3}{8}$	$\frac{6}{16}$
$\frac{7}{8} \div 2$	No	✘	$\frac{7}{16}$

3:7

When presented with a fraction divided by a whole number, children should first ask themselves, *'Is the numerator a multiple of the divisor?'*.

Provide divisions of a fraction by a whole number for children to sort into a table such as the one shown, referring back to these points:

- If the numerator is a multiple of the divisor, it can be solved with the 'unitising' method children just learnt.
- If the numerator is *not* a multiple of the divisor, the calculation must be solved by conversion to an equivalent multiplication.

$\frac{12}{15} \div 3$ $\frac{12}{15} \div 5$ $\frac{6}{7} \div 3$ $\frac{6}{7} \div 4$

$\frac{7}{18} \div 2$ $\frac{10}{11} \div 5$ $\frac{56}{65} \div 7$ $\frac{54}{56} \div 7$

Numerator is a multiple of the divisor	**Numerator is *not* a multiple of the divisor**

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3:8	Provide further practice for children. Start with calculations that can be solved using the unitising method just learnt. Once children are confident in their application of this method, progress to a selection of divisions of fractions by a whole number. Challenge children to identify the most efficient method for each calculation, and then apply it to find the solution. To promote depth of understanding, use a dòng nǎo jīn problem like the one shown opposite.	*'Find the most suitable method to solve each of these calculations.'* $\frac{12}{15} \div 4$ $\frac{14}{20} \div 7$ $\frac{9}{11} \div 3$ $\frac{20}{100} \div 5$ $\frac{5}{9} \div 4$ $\frac{16}{20} \div 2$ $\frac{8}{9} \div 3$ $\frac{10}{5} \div 5$ $\frac{12}{15} \div 5$ $\frac{7}{8} \div 6$ Dòng nǎo jīn: *'What numbers could go in the empty boxes to make these statements true? Find all the possible answers.'* $\frac{\square}{30} \div 7 = \frac{\square}{30}$

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