9 Sequences, functions and graphs

Mastery Professional Development

Solutions to exemplified key ideas

Click the heading to move to that page. Please note that these materials are principally for professional development purposes; solutions are provided to support this aim.

9.1.2.4 Understand that the solution to a linear inequality in two variables has a range of values

Understand that a line on a graph is smooth, continuous and infinitely long

Example 1:

Responses may vary but should demonstrate an understanding that:

a) Graph C meets the y-axis at 6 because it is the same equation as shown in Graph A, where the y-intercept is visible.

b) Graph A meets the x-axis at $x = (-2)$ because it is the same equation as shown in Graph B, where the x-intercept is visible.

Example 2:

a) D $(502,\ 1003)$

b) Responses may vary but should demonstrate an understanding of the additive relationship between A and B or recognition (formally or informally) that $y = 2x - 1$. For example, (6, 11), (1000, 1999) or (−2.5, −6)

Appreciate that a line on a graph delineates three distinct regions

Example 3:

Responses may vary but should demonstrate an understanding that:

- Coordinates C and D on the line must each sum to 5, but that D will have a lower x-value and a higher y-value than C.
- Coordinates A and B have the same x-value as their D coordinate (or very similar if not perceived to be in line vertically) and that A will have a slightly lower y-value than their D coordinate, whereas B will have a much higher y-value than their D coordinate. For example, A (1.5, 3.3), B (1.5, 6.8), C (4.74, 0.26) and D (1.5, 3.5).

Example 4:

a) A (red line): $x + y = 10$
B (blue line): $x + y = 11$

Responses may vary but should demonstrate an understanding that each coordinate pair on B must sum to a greater amount than each coordinate pair on A. This may include 'the y-intercept of B will be greater than the y-intercept of A' and/or 'the x-intercept of the B will be greater than the x-intercept of the A'.

b) Responses may vary but should demonstrate an understanding that the coordinates of point C satisfy the conditions that $10 < x + y < 11$ but $x + y$ is closer to 10 than 11.

Example 5:

a) Red region: $x + y < 16$
Blue region: $x + y > 20$

b) Responses may vary but should demonstrate an understanding that line B is closer to the red region than the blue. For example, $x + y = 17$.

Understand that the solution to a linear inequality in two variables has a range of values

Example 6:

a)

$y < x + 10$	$y = x + 10$	$y > x + 10$
(4, 4) *(16.3, 3.7)* *(17, 7)* *(8.9, 0.9)* *(14.3, −2.9)*		*(-1.2, 10)*

b) Responses may vary but should satisfy the relevant condition. Check by substituting x- and y-values into the equation/inequality.

9.1.3.2 Understand how to maintain equality when manipulating and combining algebraic equations

Understand that equivalence is maintained when multiplying all elements of an equation by the same amount

Example 1:

Responses may vary but should demonstrate an understanding that:

a) The line intercepts both the x-axis at 5 and the y-axis at 5.

b) Valid equations will apply the same multiplication to all elements of the equation, such as $x + y = 5$. Students may also rearrange into the form $y = mx + c$, i.e. $y = 5 - x$ or equivalent.

c) This is also representing the same relationship of the graph $x + y = 5$. Students may begin to identify that multiplying all elements of the equation by the same (non-zero) value results in no effect on the graph of the relationship.

d) This can be rewritten as $x + y = 15$.

e) This can be rewritten as $x + y = 10$.

Example 2:

a) Responses may vary but should demonstrate an understanding that:

- Mary has multiplied all elements in the first equation by 2 to get to the second equation. Mary has then multiplied all elements in the first equation by 3 (or all elements of the second by 1.5) to arrive at the third equation.
- All terms within the equation have been multiplied by the same value, preserving equality.
- There are infinite valid answers, as long as all elements of the equation are multiplied by the same value to preserve equality. Students may be likely to suggest multiplying the original equation by 4, to continue the established pattern.

b) Mary and Jay have both multiplied all elements of the equation by the same amount to arrive at a new equation. Mary has done this with an equation involving adding two terms whereas Jay has done this with an equation involving subtracting one term from another.

c) $\boldsymbol{x = 4}$ in each of Satsuki's equations and that this is down to each element of the equation being multiplied by the same amount to arrive at the next equation.

Understand that equations can be combined to create further valid equations

Example 3:

Responses may vary but should demonstrate an understanding that:

a) Combining parts or wholes from two (or more) equations and combining the original resulting parts or wholes maintains equality.

b) An equation remains balanced when the same amount is subtracted from each side. The amount can be subtracted partially from one term and partially from another, as long as they are both on the same side of the equation.

Example 4:

Responses will vary but all should demonstrate an understanding that:

a) $x = 5$ should be a solution and equality should be maintained.

b) Malcom has maintained equality by adding an equal amount to both sides for each new equation. Initially he added 3 to both sides, then x to both sides. These terms are different but equal.

c) Malcom has maintained equality by adding an equal amount to both sides. Since $x = 5$, $3x + 3$ is equal to adding $13 + x$.

d) Rita has created two valid equations by adding an equal amount to both sides on one equation, and a separate equal amount to both sides on the second equation. She has then combined all parts of both equations.

e) Rita has created a valid equation by adding the left-hand side of one equation to the right-hand side of another and vice versa, meaning that she added an equal amount to both sides of the equals sign. Since $2x = 14$, these terms can be interchangeable. The same can be said for $y + 8 = 12$.

Become proficient at combining processes to manipulate equations

Example 5:

a) A: $2x$ B: $x + 3y$ C: $2x + 2y$ D: $x + 2y$ E: $2y$ F: $\frac{1}{2}x + y$

b) C is longer by $2y$.

c) D is longer by x.

d) No, they are the same length.

e) D is twice as long as F because $2\left(\frac{1}{2}x + y\right) = x + 2y$.

Understand that equations can be manipulated, combined and compared to create further valid equations

Example 6:

a) $e + f = 3 + 5 = 8$

b) (i) $(g + h) + (j + k) = 3 + 5 = 8$ (ii) $(j + g) + (k + h) = g + h + j + k = 3 + 5 = 8$

c) $s + t + u + v + 4mn = 3 + 5 = 8$

d) Responses may vary but should demonstrate a correct expression equal to 8. For example, $2x + 3y + x + y$ or $3x + 4y$

Example 7:

a) (i) $e - f = 3 - 5 = (-2)$ (ii) $f - e = 5 - 3 = 2$

b) (i) $(g + h) - (j + k) = 3 - 5 = (-2)$ (ii) $(j + k) - (g + h) = 5 - 3 = 2$

c) $(s + t + u + v) - 4mn = 3 - 5 = (-2)$

d) Responses may vary but should be:

(i) An expression with a value of 2. For example, $(x + y) - (2x + 3y)$.

(ii) An expression with a value of -2. For example, $(2x + 3y) - (x + y)$.

Example 8:

a) (i) $(4m + 5y) + (4m + 3y) = 33 + 31 = 64 = 8m + 8y$

(ii) $(4m + 5y) - (4m + 3y) = 33 - 31 = 2 = 2y$

(iii) $(5m - 3y) + (4m + 3y) = 32 + 31 = 63 = 9m$

b) Responses may vary but should demonstrate an understanding that:

in (i) no terms are eliminated; in (ii) m is eliminated; and in (iii) y is eliminated.

c) Part a (ii). As the m has been eliminated, we are left with $2 = 2y$ which can be solved.

d) Part a (iii). As the y has been eliminated, we are left with $63 = 9m$ which can be solved.

9.1.3.4 Appreciate that linear simultaneous equations can be solved by elimination or substitution

Understand how adding or subtracting two equations can result in one of the variables being eliminated

Example 1:

a) $\mathbf{19 + 3 = 22}$

b) Responses may vary but could include $\mathbf{4x}$ or $\mathbf{3x + y + x - x}$.

c) $\mathbf{x = 5.5}$

d) $\mathbf{y = 2.5}$

Example 2:

a) Yes, Ella is correct. The difference between the two orders is one tea and the difference in the cost is 22 p.

b) Representations may vary but could include appropriate models or algebraic representations such as $x + y = 70$ and $x - y = 58$.

c) The speed of the swordfish is 64 km/h. The speed of the current is 6 km/h.

Example 3:

a) 10 cm

b) 7 cm

c) x = 3 cm, y = 1 cm, B = 6 cm, D = 5 cm, E = 2 cm, F = 2.5 cm

d) x = 8 cm, y = 5 cm, A = 16 cm, B = 23 cm, C = 26 cm, D = 18 cm

Example 4:

Responses may vary but should demonstrate an understanding that:

a) The length of A is $2x + y$ because the top row, with a value of A, is equal in length to the bottom row, with a value of $2x + y$.

b) Since B $+ y = 2x$, the length of B is $2x - y$. The bottom row shows $2x$. Subtracting the y shown in the top row from that length gives the length of B.

c) The total length of the top row is equal to the total length of the bottom row. The length of A, B and y is the same as the length of $4x$ and y, so $A + B$ has the same length as $4x$.

d) $2x + y$ = A. $2x - y$ = B. Therefore $A + B = 4x$.

e) $2x + y$ = A. $2x - y$ = B. Therefore $A - B = 2y$.

f) A has the same length as B $+ 2y$, therefore $A = B + 2y$. It follows that if you were to subtract B from A it would be $2y$.

Example 5:

Responses may vary but should demonstrate an understanding that:

a) The two rows of the top bar model are equal in length, and so demonstrate that $7 = 4x + y$. By the same logic, the bottom bar model shows $2 + y = 2x$. The bottom row can also be read as $2x - y = 2$ by considering $2x$ as the whole and 2 and y as constituent parts.

b) By placing the two bar models next to each other horizontally, the top row now represents $7 + 2 + y$ and the bottom row now represents $4x + y + 2x$. Simplifying this leads to $9 + y = 6x + y$, so $6x$ must be equivalent to 9. Or, rearranging both bars so the y is on the left/right hand side, I can see two equal lengths of 9 and $6x$.

c) As $6x = 9$ only has one unknown, we can solve this equation. We can then substitute x with this value to find the value of y. This could also be demonstrated using the bar model using equal lengths.

Example 6:

Responses may vary but should demonstrate an understanding that:

a) (i) The top row of diagram 1 represents 11 and the bottom row represents $2x + y$. They are equal in length, showing that $2x + y = 11$. The top row of diagram 2 represents 13 and the bottom row represents $x + 3y$. They are the same length, showing that $x + 3y = 13$.

(ii) By combining the two bar models, the combined top row represents $11 + 13 = 24$ and the bottom row represents $(2x + y) + (x + 3y)$. As the bars are the same length, this shows that $(2x + y) + (x + 3y) = 24$.

(iii) $(2x + y) + (x + 3y)$ can be simplified to $3x + 4y$.

(iv) As $2x + y = 11$ and $x + 3y = 13$, $(2x + y) - (x + 3y) = 11 - 13 = (-2)$.

(v) $(2x + y) - (x + 3y)$ can be simplified to $x - 2y$.

(vi) There is more than one unknown variable in each equation.

b) (i) The top bar of diagram 1 represents 11 and the bottom bar represents $2x + y$. They are equal in length, showing that $2x + y = 11$. By collecting like terms for diagram 2, the top bar represents 26 and the bottom bar represents $2x + 6y$. They are equal in length, showing that $2x + 6y = 26$.

(ii) By combining the two bar models, the top bar represents $11 + 13 + 13$ which is equal to 37 and the bottom bar represents $(2x + y) + (2x + 6y)$. As the bars are the same length, this shows $(2x + y) + (2x + 6y) = 37$.

(iii) $(2x + y) + (2x + 6y)$ can be simplified to $4x + 7y$.

(iv) As $2x + y = 11$ and $2x + 6y = 26$, so $(2x + y) - (2x + 6y) = 11 - 26 = (-15)$.

(v) $(2x + y) - (2x + 6y) = (-5y)$

c) $x = 4, y = 3$ Part (v) helped because there was an equation with only one unknown.

Understand how substituting one expression for another can be used to solve simultaneous equations

Example 7:

a) Responses may vary and the mathematics may be arrived at using different representations. One example solution is given below:
Using m for the number of £2 coins and t for the number of tokens:

	$m + t$	=	30
substituting	m	=	$\frac{52}{2}$
gives	$\frac{52}{2} + t$	=	30
and so	t	=	4

b) x and y could represent the number of times a £5 has been changed and the number of times a £2 coin has been changed. These could be done either way around at this stage.

c) x

d) You could substitute y for $(45 - x)$ in Ken's equation.

e) 15 were £5 notes. 30 were £2 coins.

Example 8:

Responses may vary but should demonstrate an understanding that:

a) The top bar of diagram 1 represents 45 and the bottom bar represents $2x + y$. They are the same length, showing that $2x + y = 45$.
The top bar of diagram 2 represents 24 and the bottom bar represents $x + y$. They are the same length, showing that $x + y = 24$.

b) The $x + y$ section of the bottom bar on diagram 1 can be substituted with 24 as we know $x + y = 24$. 45 is still x more than 24, so this diagram shows $x + 24 = 45$.

c) $x = 21, y = 3$

Understand that elimination and substitution are equally valid methods for solving simultaneous equations

Example 9:

a) Using Eli's: $2x + 1 = 11$ so $x = 5$.

b) Using Sarah's: $y = 11 - 2 \times 5$ so $y = 1$.

c) Responses may vary but should demonstrate an understanding that both approaches are valid and will lead to the same solution.

d) Responses may vary but should demonstrate an understanding that Eli's method may be easier than Sarah's, when the coefficients mean that one of the variables can easily be eliminated and when the variables are on the same side of the equation in both cases. Sarah's may be easier when the equation can easily be rearranged to isolate one of the variables.

Example 10:

Responses may vary for part (ii) but should demonstrate the understanding outlined below:

a) (i) Elimination.

(ii) The coefficient of x is the same in both equations, meaning they can be eliminated without the need for additional work.

b) (i) Substitution.

(ii) y is already the subject of the first equation and so can be substituted without additional work from the second equation.

c) (i) This will be personal choice for the solver.

(ii) Both approaches require one step of rearranging initially.

d) (i) Substitution.

(ii) We can make p the subject of the formula in the second question with one step.

9.1.3.5 Represent and interpret the solution to linear simultaneous equations

Understand that a solution to a pair of simultaneous equations must satisfy both at the same time

Example 1:

Responses may vary but should demonstrate an understanding that:

a) Yes, the second number is 7 for both Gaelen and Jennika.

b) The sixth number will be when they are both on the same square at the same time.

Example 2:

Yes. $\boldsymbol{x = 20}$, $\boldsymbol{y = 10}$

Example 3:

a) For $y + 2x = 12$ and $y - 2x = 0$: $x = 3, \quad y = 6$

For $y + 2x = 12$ and $x + y = 6$: $x = 6, \quad y = 0$

For $y - 2x = 0$ and $x + y = 6$: $x = 2, \quad y = 4$

b) Responses may vary but should demonstrate an understanding that it is not possible. These are non-concurrent linear equations; each pair only has one set of solutions and none of these solutions are common between different pairs.

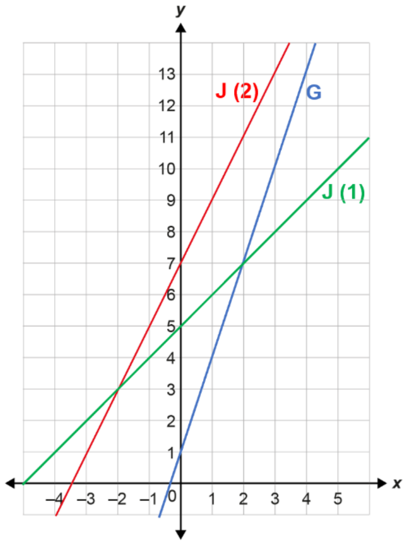
c) Responses may vary but should demonstrate an understanding that each pair of equations intersect each other at one point. Each point is different between each pair of equations.

Example 4:

Responses to parts a and c may vary but should demonstrate the understanding outlined below:

a) Part b from *Example 1* shows the same relationship where Jennika started on square 7 and moved forward 2 squares per turn.

b) Gaelen's second turn, G, is represented by the graph of $y = 3x + 1$; Jennika's first rule, J(1), is represented by the graph of $y = x + 5$; Jennika's second rule, J(2), is represented by the graph of $y = 2x + 7$.



c) The lines showing the same relationship as Gaelen and Jennika in the first instance intersect at (2,7), which supports the answer to *Example 1* part a (i.e. that Gaelen and Jennika's second number would have been 7 for both). It also demonstrates that if Gaelen and Jennika were to be on the same square for *Example 1* part b, it wouldn't be until they are past number 13.

Interpret graphically whether a pair of simultaneous linear equations will have one, none or an infinite number of solutions

Example 5:

Responses to some parts may vary but should demonstrate the understanding outlined below:

a) A because when $x = 0, y = 30$.

b) The lines appear to be parallel so line B might be $y = 17x + 2.5$ (Gradient the same, y-intercept between 2 and 4). Or line B might have a slightly greater gradient, so could be $y = 17.5x + 2.5$.

c) The lines representing two linear equations with different gradients will have one point of interception.

d) The lines will meet further up the axes.

e) Any equation with $y = 17x + c$ (where c is any number except 30).

f) You may find that $0 = 30 - c$, which shows that either the equation has no solution (in the case that $c \neq 30$ and the statement is incorrect), or that $c = 30$ which means the lines are the same.

9.2.1.3 Recognise and interpret geometric growth in a sequence

Begin to appreciate the increasing/decreasing difference between terms in a geometric sequence

Example 1:

a) Responses may vary but students might assume that option A is better initially, until they notice that for option B they receive more money than option A on the 18th day (£131 072 vs £18 000) and significantly more for each day that follows.

b) See above.

c) February: $2^{27} = £134\,217\,728$. March: $2^{30} = £1\,073\,741\,824$. Difference: $£939\,524\,096$

Example 2:

a) 1

b) 32 768

c) Row 3

d) Row 4

e) Row 4 for both

Example 3:

a) Responses will vary depending on students' initial choice of line length (x) but should demonstrate an understanding that line 1 = x units, line 2 = 1.1x units, line 3 = 1.21x units, line 4 = 1.331x units and so on. Line 10 would have a length of 2.5937x units.

b) Responses may vary but should demonstrate an understanding that Bela is wrong. Each new line is 10% longer than the previous line, not 10% longer than the original line. Each 10% being added on is worth more than the previous 10%.

c) Responses may vary but should demonstrate an understanding that Bela's tenth line will be much smaller than her original line, but it will exist. If line 1 is x units, line 10 will be 0.3487x units.

Example 4:

Responses may vary but should demonstrate an understanding that nobody will finish the cake, though the slices will become increasingly more difficult to cut. All sequences where the multiplier is between 0 and 1 will tend towards, but not meet, 0.

Understand that geometric growth can be thought of as repeated multiplication.

Example 5:

a) 1, 2, ***4***, ***8***, 16, ***32***

b) 1, ***10***, ***100***, 1 000, ***10 000***

c) 1, ***6***, 36, ***216***, ***1 296***

d) ***1***, ***4***, ***16***, ***64***, 254, 1024

Example 6:

a) 999 000

b) 999 000 000 000

Appreciate the difference between linear, non-linear and geometric sequences.

Example 7:

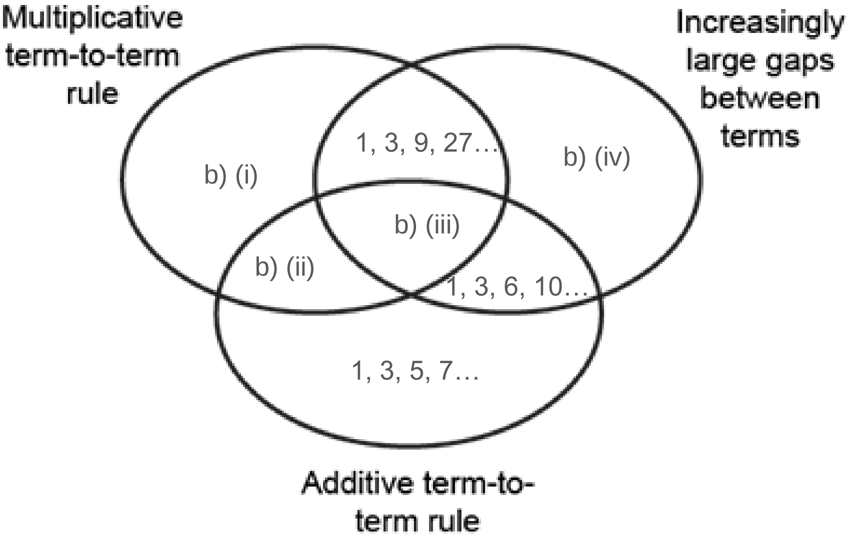
a) Responses may vary, as there are infinitely many quadratic sequences with 2 given terms, but 49 is a likely student response.

b) 47

c) Responses may vary, for example 51.84.

Example 8:

a)



b) Using the labelling in the solution above, responses may vary but should demonstrate an understanding that:

(i) Any sequence with a multiplier between 0 and 1 will be acceptable.

(ii) It is not possible for a term-to-term rule to be both multiplicative and additive.

(iii) It is not possible for a term-to-term rule to be both multiplicative and additive.

(iv) Any sequence (a succession of terms formed according to a rule) with increasingly large gaps between terms that is neither additive nor multiplicative. For example, 1, 4, 9, 16... (additive **pattern**, not additive **rule**).

Example 9:

Responses may vary but should demonstrate an understanding that, except for the growth between the first and second number, $5^n + 1$ grows the fastest as it is growing exponentially.

9.2.2.3 Understand and use method(s) to express the nth term of a quadratic sequence

Identify growth in quadratic sequences *Example 1:* a) 21 b) 47 c) $2n + 1$
Example 2: a) Responses may vary but should demonstrate an understanding that Dan is incorrect because there is a constant second difference of $+2$ (although students might not be referring to it as second difference at this stage). b) Responses may vary but should demonstrate an understanding that Effie is correct. c) Next difference $+13$, producing the 7th square number: $36 + 13 = 49 = 7^2$. Next difference $+ 15$, producing the 8th square number: $49 + 15 = 64 = 8^2$.
Example 3: a) 16 and 25 b) 36 and 49 c) 121 and 144 d) 256 and 289
Appreciate that, in a quadratic sequence, the second difference is constant, and is double the coefficient of n^2 *Example 4:* a) 42 b) 94 c) $4n + 2$
Example 5: Responses may vary but should demonstrate an understanding that Eddie is correct.
Example 6: a) 6, 24, 54 b) $a = 5$ c) 16
Example 7: a) $m = 3$ b) $m = 5$

Example 8:

Responses may vary but should demonstrate an understanding that:

a) The coefficient of *a* is growing by 1 each time and is equal to the term number. The value of *b* remains constant.

b) From Logan's counters, the second term is represented by $2a + b$. Izzy's second term is 190. Logan has shown that these are equal.

c) For example:

$a + b$	=	153
$3a + b$	=	227
$4a + b$	=	264
$5a + b$	=	301

d) $a = 37$, $b = 116$

e) Using the difference between terms, a, to generate which multiples we are using leads to an initial sequence of a, $2a$, $3a$. To achieve Logan's sequence, we need to add b. The *n*th term is $an + b$ which matches each equation in part *c* whereby *n* in the term number.

Example 9:

Responses may vary but should demonstrate an understanding that:

a) The a counters form squares with a side length that grows by 1 counter per term; the b counters form a line that grows by 1 counter per term; the c counters do not change.

b) The coefficient of a is the square of the term number; the coefficient of b is the term number; the coefficient of c is 1 each time. The coefficients off a, b, and c are respectively n^2, n and 1.

c) The coefficients of a, b and c are as described in part b. These expressions are then made equal to each corresponding term from Izzy's sequence to form the equations.

d) They may subtract the difference between terms 1 and 2 from the differences between terms 2 and 3 which will eliminate b. A further equation of $2a = 6$ would be formed.

e) Logan is correct. As we know that $a = 3$, we can subtract $3n^2$ from each term in Izzy's sequence. We would then be left with $bn + c$.

Appreciate that all quadratic sequences can be considered as a combination of a quadratic and linear sequence

Example 10:

a) 9, 11, 13, 15, 17

b) 3, 12, 27, 48, 75

c) 3

d) $3n^2 + 2n + 7$

Example 11:

Responses may vary but should demonstrate an understanding that:

a) There is no growth in the striped, yellow tile. It is constant.

b) There is another row with three more spotted, green tiles each time. This is an example of linear growth.

c) There are two more columns and one more row of blue tiles. This could also be seen as two squares that increase in side length by one each time. This is an example of quadratic growth.

d) Each image has three more spotted, green tiles than the previous one. The three extra spotted, green tiles are because of an additional row each time.

e) For the first pattern, there are two 1 × 1 squares. For the second pattern, there are two 2 × 2 squares next to each other. For the third pattern, there are two 3 × 3 squares next to each other. For the fourth pattern, there are two 4 × 4 squares next to each other.

f) Plain blue: $2n^2$. Spotted green: $3n$. Striped yellow-and-black: 1. All tiles: $2n^2 + 3n + 1$.

Example 12:

a) Students' approaches will vary based upon their preferred method. One possible approach, using $an^2 + bn + c,$ is given below:

Term 1 $a + b + c = 0$

Term 2 $4a + 2b + c = 3$

Term 3 $9a + 3b + c = 10$

Difference between terms 1 and 2 is $3a + b = 3$

Difference between terms 2 and 3 is $5a + b = 7$

The second difference algebraically $2a = 4$

$a = 2$

Substituting into the differences between terms 1 and 2 $3a + b = 3$

$6 + b = 3$

$b = (-3)$

Substituting into term 1 $2 + (-3) + c = 0$

$c = 1$

General term $2n^2 - 3n + 1$

Responses to parts b) and c) may vary but should demonstrate an understanding that:

- Caitlin has worked out the difference between each term and then the second difference, which was 4.
- She halved the second difference to find the coefficient of n^2, which was 2.
- She worked out the difference between the sequence and the terms generated using $2n^2$.
- What was missing from the original sequence was a linear sequence. Catilin worked out the general term of this linear sequence, which was $(-3)n + 1$. Finally, Catlin combined $2n^2$ and $(-3)n + 1$ to arrive at $2n^2 - 3n + 1$.

Appreciate the difference between growth in a polynomial and geometric sequences

Example 13:

a) Responses may vary but should demonstrate an understanding that n^2 has a constant difference on the second row of differences. n^4 has a constant difference on the fourth row of differences. For n^t the constant difference happens on row t.

$\boldsymbol{n^2}$	**1**		**4**		**9**		**16**		**25**
1st difference		3		5		7		9	
2nd difference			2		2		2		

$\boldsymbol{n^4}$	**1**		**16**		**81**		**256**		**625**		**1296**		**2401**
1st difference		15		65		175		369		671		1105	
2nd difference			50		110		194		302		434		
3rd difference				60		84		108		132			
4th difference					24		24		24				

b) The constant difference happens on the third row of differences.

c) The constant difference happens on the second row of differences.

d) (i) The constant difference would happen on the fifth row of differences.

(ii) The constant difference would happen on the eighth row of differences.

(iii) The constant difference would happen on the twenty-fifth row of differences.

e) The index number increases for each term. The row where the constant difference occurs is equal to the largest index number in the expression. As this increases each time the rate of change will continually vary.

9.3 Exploring quadratic equations, inequalities and graphs

9.3.1.1 Understand that all quadratics can be written in the form $a(x-h)^2+k$ (completing the square)

Understand that any square can be deconstructed into two smaller squares and two congruent rectangles

Example 1:

a) Red diagonally-striped square: 10 × 10; blue striped rectangles:10 × 4; yellow square: 4 × 4.

b) Red diagonally-striped square: 25 × 25; blue striped rectangles: 25 × 4; yellow square: 4 × 4.

c) Red diagonally-striped square: 10 × 10; blue striped rectangles: 10 ×1.6; yellow square: 1.6 × 1.6.

Example 2:

a) 4 plain dark grey tiles. The pale grey tiles are arranged in a 10 × 10 array. There are two congruent rectangles of striped tiles, each using 20 tiles in 10 × 2 arrays. This means that the dark grey tiles are arranged in a 2 × 2 array.

b) 20 plain dark grey tiles. He can add an additional two rows/columns of 10 to each of the striped tiles. The dark grey tiles would go from a 4 × 4 array to a 6 × 6 array. 10 striped tiles would remain.

Example 3:

a) Responses may vary but should demonstrate an understanding that both are correct.

b) Responses may vary but might include that Yukiko has found the area of each section of the square and added them together to find the total area, and Xavier has found the total side length and squared it to find the area of the square.

c) Yukiko: $(8\times 8)+(2\times 3\times 8)+(3\times 3)$ Xavier: $(8+3)^2$

Example 4:

a) Louie could arrange them into 2 × 2 squares. He would have 25 of them.

b) There would be no tiles left over.

c) He could have six 4 × 4 squares. He would have 4 tiles left over.

d) Louie could have two 7×7 squares. He would have 2 tiles left over.

e) Responses may vary but should demonstrate an understanding that there would be 50 tiles available for each square. An 8 × 8 square would require 64 tiles for each square.

Notice that any value or expression can be written as a square ± an adjustment

Example 5:

a) Responses may vary but should indicate that since 534 is between 23^2 and 24^2, the tiles cannot be arranged into a square.

b) 23×23 . This will require 529 tiles. There will be 5 left over.

c) 1 442 would need to be a square number.

d) Della would create a 38 × 38 square but with 2 tiles missing because 1 442 is two less than 38^2.

e) The dimensions of the square will be 37 × 37. This will require 1 369 tiles. There will be 73 tiles left over.

f) They could be arranged into an 8 × 8 square and a 3 × 3 square with no tiles left over.

Example 6:

Responses may vary but should demonstrate an understanding that:

a) (i) Not possible; the smallest difference between two square numbers is 3, and there are 2 tiles left.

(ii) Possible if $x = 5$. The difference between 5^2 and 6^2 is 11 .

(iii) Always possible. The term inside the bracket will be squared, always producing a square number.

(iv) Not possible, because $(x + 2)^2$ is a square number and there are not any consecutive square numbers with a difference of 6.

b) (i) Yes. You could make an $x \times x$ square with 2 tiles left over.

(iv) Yes, for $(x + 2)^2 + 6$, the $(x + 2)^2$ term will always form a square and there will be 6 tiles left over. In general, any expression of the form $ax^2 + bx + c$ can be written as a square with an adjustment, (i.e. in the form $m(x \pm p)^2 \pm q$) which is an essential understanding when completing the square.

Connect the algebraic and pictorial representations of completed squares

Example 7:

Red square: $x \times x$
Blue striped rectangles: $x \times 4$
Yellow square: 4×4

Example 8:

a) Responses may vary but should demonstrate an understanding that $(x + 6)$ represents the side length of the square, therefore $(x + 6) \times (x + 6)$ is the area.

b) Responses may vary but should demonstrate an understanding that x^2 is the area of the top-left grey square. $12x = 6x + 6x$, the areas of the two stripey rectangles. 36 is the area of the bottom-right grey square.

c) Yes. Students' justifications may vary but should show that they understand how to expand and simplify $(x + 6)^2$.

d) Yes. Students' justifications may vary but should demonstrate that, since 1 has been added to both sides of the equation in part c, equality has been maintained.

Example 9:

a) (i) $x^2 + 2x + 1$

(ii) $x^2 + 4x + 4$

(iii) $x^2 + 6x + 9$

b) Responses may vary but should demonstrate an understanding that the coefficient of x is always the constant in the bracket multiplied by two.

c) The next three in the sequence are: $x^2 + 8x + 16$, $x^2 + 10x + 25$ and $x^2 + 12x + 36$.

d) Responses may vary but should demonstrate an understanding that it would be $x^2 + 2ax + a^2$.

e) (i) $x^2 - 2x + 1$

(ii) $x^2 - 4x + 4$

(iii) $x^2 - 6x + 9$

f) Responses may vary but should demonstrate an understanding that the coefficient of x^2 is always 1, the coefficient of x is always the constant in the bracket multiplied by two and the numerical term

is always the constant in the bracket squared. The x^2 and numerical terms in part a match their corresponding expressions in part e, as $(-x)^2 = x^2$. However, the different signs do have an effect on the x terms: the coefficients of the x term are positive in part a but negative in part e.
Example 10: a) (i) $x^2 + 2x$ (ii) $x^2 - 4x$ (iii) $x^2 + 6x$ (iv) $x^2 - 14x$ (v) $x^2 + 24x$ b) Responses may vary but should demonstrate an understanding that the subtractions at the end have eliminated the numerical term from the expansion. c) (i) $(x+5)^2 - 25$ (ii) $(x+10)^2 - 100$ (iii) $(x+1000)^2 - 1\,000\,000$ (iv) $\left(x + \frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2$ or $\left(x + \frac{a}{2}\right)^2 - \frac{a^2}{4}$
Example 11: Responses may vary, but should indicate an understanding that equations A, B and C are also true and can be checked by comparing numerical terms, while D, E and F do not follow from the original statement.

9.3.1.4 Understand that an equation written in the form $(ax + b)(cx + d) = 0$ can be solved

Understand that, for the product of two numbers to be zero, at least one of them must be zero *Example 1:* Responses may very but should indicate an understanding that: a) The area increases. b) The area decreases to zero. c) When $0 \leq x < 3$, or when $7 < x \leq 10$, one side length will be negative, resulting in a negative area. When $x = 3\ or\ 7$, one side length will be 0, resulting in no area. When $3 < x < 7$, the area is positive.
Example 2: Responses may vary but should demonstrate an understanding that the relevant statement is: a) Always true. Any number multiplied by zero is zero. b) Always true. Any number multiplied by zero is zero. c) Sometimes true. If at least one of x or y is zero. d) Sometimes true. If $x = 0$. e) Sometimes true. If $a = 6$ and/or $b = -1$. f) Sometimes true. If $a = 6$ or $a = -1$.

g) Never true: m and n are squared, so will always be zero or positive. They then have 1 added to them, so each bracket will always be greater than 0.

Understand that, for the product of two expressions to be zero, at least one of them must be zero

Example 3:

a) $x = 8$ b) $x = 8$ c) $x = 4$

d) $x = 8$ e) $x = 0$ or $x = 8$ f) $x = 2$ or $x = 8$

Example 4:

a) Yes, $x = 0$ b) No c) No d) Yes, $x = -4$

Example 5:

Responses may vary but should demonstrate an understanding that:

a) Melanie is not correct, because when multiplying 1 by a the answer is a, not 1.

b) Asif is not correct because there are other possible values.

c) Seema is partially correct that either $(1 - 2x)$ or $(4 + 3x)$ must be zero, but it is not because $0 \times 0 = 0$, it is because $0 \times$ anything $= 0$.

Example 6:

a) $x = 6$

b) Responses may vary but should include an understanding that knowing two numbers multiply to make 3 does not tell us what each number is directly. Students may consider factors of 3, noticing that $1 \times 3 = 3$ and considering values of 1 and 3 for the terms in brackets, leading to x values of 6 and 4, as above since $(7 - 4)(4 - 3) = 3$ and $(7 - 6)(6 - 3) = 3$.

c) $x = 8$

d) Responses may vary but students might consider factors of 5, leading to possible values of -1×5, and 1×-5, this leads to $x = 2$ and $x = 8$ as above.

e) $x = 7$

f) Responses may vary but should demonstrate an understanding that either the height or width must be zero to give an area of zero.

9.3.2.5 Understand the connection between the graphical and algebraic interpretations of roots and intersections

Understand that the coordinates of a point on a curve represent a solution pair to the equation that defines the curve

Example 1:

a) 4

b) Responses may vary but should demonstrate an understanding that the graph can be used to read the corresponding value of y.

c) $\frac{1}{2} \times (-\mathbf{4})^2 + (3 \times -\mathbf{4}) - 4 = -8$

d) $x = 2$

Recognise the importance of zero when working with graphs of quadratic functions

Example 2:

a) $x = 0$ *or* $x = -2$ b) $x = -1$ c) $x = 2$ *or* $x = -4$

Example 3:

Responses for a, b and c may vary as they are estimated from the graph.

a) $x = -30, x = 26$ b) $x = -17, x = 13$ c) $x = -16, x = 12$

d) Responses may vary but should demonstrate an understanding that the grid is difficult to read accurately given the scale used. Values chosen can be substituted into the equation to inform of accuracy.

e) Responses may vary but should demonstrate an understanding that $(x + 16)(x - 12) = 0$ is the easiest to write an answer for because either $(x + 16) = 0$ or $(x - 12) = 0$.

Example 4:

Responses may vary but should demonstrate an understanding that, because the x-axis is the line $y = 0$, we know that both of those points lie on the line $y = 0$. Therefore, we are looking to solve the equation $0 = (x + 23)(x - 3)$ and, because it is written in factorised form, can easily make one of the brackets equal to 0 by substituting either −23 or 3. A is (−23, 0) and B is (3, 0).

9.3.3.1 Understand that a quadratic splits a plane into three regions, and be able to define them

Understand that the whole line can be considered as points on the curve

Example 1:

Responses may vary but should demonstrate an understanding that:

a) Neither is correct, because there are an infinite number of points on a graph. They have both labelled points that are on the intersection of gridlines only. Chima has either used graph paper with smaller squares or has used a different scale, resulting in more points being marked.

b) Freddie's graph doesn't have gridlines or specific points marked, although it is the same graph showing the same infinite set of points.

c) By substituting $x = 1\,000$ into the equation of the line. Zippy means that this is a point on the curve.

Appreciate that a line graph identifies three distinct regions

Example 2:

a)

$\boldsymbol{y < x^2 + 3x + 1}$	A and E
$\boldsymbol{y = x^2 + 3x + 1}$	D
$\boldsymbol{y > x^2 + 3x + 1}$	B and C

b) Responses may vary. Any coordinates that satisfy the given equation or inequality.

c)

	$y < x^2 + 2x + 4$	$y = x^2 + 2x + 4$	$y > x^2 + 2x + 4$
$y < x^2 + 3x + 1$	A and E	-	-
$y = x^2 + 3x + 1$	D	-	-
$y > x^2 + 3x + 1$	B	C	-

d) Responses may vary. Any coordinates that satisfy both of the given equations or inequalities.

9.3.4.2 Understand that there are either zero, one or two solutions to a set of simultaneous equations where one is linear and the other is quadratic

Appreciate that there can be up to two solutions to a pair of simultaneous equations where one is linear, and the other is quadratic

Example 1:

a)

$y < x^2 + 3x - 6$	A and E
$y = x^2 + 3x - 6$	B and D
$y > x^2 + 3x - 6$	C

b)

	$y < 2x - 4$	$y = 2x - 4$	$y > 2x - 4$
$y < x^2 + 3x + 1$		A and E	
$y = x^2 + 3x + 1$		B and D	
$y > x^2 + 3x + 1$		C	

c) Responses may vary but should demonstrate an understanding that, because $y = 2x - 4$ is a straight line, there will be no other coordinates that are common to both lines. The linear graph and quadratic graph will not intersect more than twice.

Example 2:

a) −1.2

b) (i) and (ii) are both −6.8

c) Responses may vary but should demonstrate an understanding that there are two coordinates which lie on both $y = 2x - 8$ and $y = x^2 - 2x - 6$. There are two points of intersection.

Example 3:

a) (0, −6) and (4, 2)

b) Responses may vary but should demonstrate an understanding that, while there can be two pairs of solutions, they disagree because the linear graph may go underneath the quadratic graph and not

touch at all, or it may touch only once. (Note: based on the structure of these examples building up towards the key idea, students may not be considering zero solutions or one solution at this stage.)
Example 4: Responses may vary but should demonstrate an understanding that: a) It is a solution to both if it is where the two lines intersect. b) There will be either zero, one or two solutions. (Note: based on the structure of these examples building up towards the key idea, students may not be considering zero solutions at this stage.)
Appreciate that it is possible for there be no solutions to a pair of simultaneous equations where one is linear, and the other is quadratic *Example 5:* Responses may vary but should demonstrate an understanding that the blue line does not cross the quadratic graph, so there are no instances of them having the same coordinates. Therefore, there are no real solutions.
Example 6: a) $y = -x - 5$ and $y = x^2 - 5x - 1$ b) $y = 3x - 17$ and $y = x^2 - 5x - 1$ c) $x = 0,\ y = -1$ and $x = 6,\ \ y = 5$ d) No real solutions

9.4.1.1 Connect a graphical representation with a real-life context (including kinematics)

Connect points on graphs with given contextual information

Example 1:

a) Responses may vary but should demonstrate an understanding that Thomas used the y-axis to show age because his twin siblings have the same age, represented by points D and E having the same y-value. Therefore, the x-axis shows height.

b) Taller

c) A = Mum, B = Thomas, C = Stu

Example 2:

Assumptions may be made that the x-axis represents the size of the carton, and the y-axis represents the price, since A and E are aligned vertically, as are C and D, and it is more likely that there would be unique values for the price than the size.

a) Responses may vary but could demonstrate an understanding that:

(i) F is the most expensive, independently of the choice of axes.

(ii) C and F are the best value under the assumptions above because they are much larger than B, G, E and A but the price increase isn't as steep. D is the same price as C.

(iii) A and E or C and D.

b) A and D are likely to be the organic milk because they are the same size as E and C respectively, but more expensive.

c) Responses may vary but might include that B and G represent the worst value for money.

d) Point C and point F are directly proportional to one another. This tells us that they have the same rate of change and therefore represent the same cost per unit of milk. The same can be said for point B and G.

Example 3:

a) Responses may vary but should demonstrate an understanding that the steeper line (blue) represents the journey by road. The shallower line (red) represents the footbridge.

b) Approximately 6.5 times

c) Walking

Connect the shape of graphs with given contextual information

Example 4:

a) C b) A c) D d) B e) E

Example 5:

a) Graph B

b) Responses may vary but should demonstrate an understanding that the runner was faster coming down the ladder. The second near-horizontal part of the graph is shorter.

Use appropriate calculations to find quantitative information from a graph

Example 6:

a) Responses may vary but should demonstrate an understanding that Vicky is looking at the steepness of each section. Between the 2nd and 3rd blue points, the graph is least steep.

b) speed $= \frac{distance}{time} = \frac{16}{12}$

c) Responses may vary but should demonstrate an understanding that the gradient here is the rate of change of distance with respect to time, the same calculation as $\frac{change\ in\ distance}{change\ in\ time}$.

Example 7:

a) Speed
b) Distance
c) Speed
d) Acceleration
e) Speed or acceleration
f) Distance

Example 8:

a) (i) 4 m/s (ii) 7 m/s (iii) 73 m

b) (i) 0 m/s (ii) 2 m/s (iii) 10 m

9.4.1.4 Interpret the graph of a function as a representation of the mapping of the domain onto the range

Begin to use the language of 'function' to connect two variables

Example 1:

a) '... how full of water the kettle is.'

b) Any phrase referencing a relevant variable, for example: '... the starting temperature of the water', or '… the reliability of the power supply'.

c) Any phrase referencing an irrelevant variable, for example: '... how old Jen is', or '… whether Jen is watching the kettle boil'.

d) Any examples of real-life functions. For example, the amount of fuel used by a car as a function of the distance travelled.

Example 2:

a) '… the number of gold stars.'

b) '... good effort.'

c) Responses may vary but students may state that as house points is a function of gold stars, and gold stars is a function of good effort, we could also state that, 'The number of house points is a function of good effort'.

Example 3:

a) Any four numbers whereby they can be rounded to 0 with any degree of accuracy.

b) Any four numbers whereby they can be rounded to 1 with any degree of accuracy.

c) Responses may vary but should demonstrate an understanding that Aaravi's function machine doesn't clearly show that there is an infinite number of possible inputs.

Example 4:

a) Responses may vary but should demonstrate an understanding that Geri has used function notation. It is stating that the value of the function when evaluated at x can be found be considering if x is less than 0.5, in which case the output/domain would be 0, or if x is greater than or equal to 0.5, in which case the output/domain would be 1.

b) $f(x) = \begin{cases} 1, x < 1.5 \\ 2, x \geq 1.5 \end{cases}$

c) Responses may vary but should demonstrate an understanding that Geri's representation covers all possible inputs/domains and outputs/ranges better than the representations in *Example 3*.

Example 5:

a) Responses are likely to vary, but may demonstrate an understanding that:

- All represent the height of the swing being a function of the time elapsed.
- The function machines are providing limited information. For example:
  - The first function machine does not show any pairs of values but does show the two variables.
  - The second function machine shows five values that map to 0.4, so does not include any other times that map to that height, or any other possible swing height.
- The coordinate table is providing some pairs of values but does not provide any of the infinite pairs of values that exist between each coordinate pair.
- The graph is showing height and time co-varying at the same time.

b) Responses may vary but should demonstrate an understanding that the final representation, the graph, best models the relationship between height and time as the values of the input/domain and output/range co-vary simultaneously. The height of the swing is a function of the time elapsed, but both values are continuously changing. The other representations don't demonstrate this continuous relationship clearly.

Example 6:

a) Responses may vary but some key features of likely responses are outlined below:

(i) *Each value of the input maps onto a single output value:* demonstrated by the **graph** (for every value on the x-axis, there is a single corresponding value on the y-axis) and **table of values** (for every x-value in the table, there is a single row of y-values).

(ii) *More than one input value maps onto each output value:* demonstrated by the **graph** (the parabolic shape of the graph), **table of values** and **coordinate pairs** ($y = 4$,for example, can be seen as the output for both $x = -2$ and $x = 2$).

(iii) *When $x = 0$, $y = 0$:* demonstrated by the **graph** (curve goes through the origin), **table of values** and **coordinate pairs** (sight of (0, 0) either as coordinates or within the table).

(iv) *As the value of x increases, the change in y-value depends on what the value of x was:* demonstrated by the **graph** (the rate of change/gradient of the curve is continually changing) **coordinate pairs** and **table of values** (the difference between each consecutive y-value changes each time).

(v) *There are infinite sets of points that satisfy this function:* demonstrated by the **graph** (the smooth, continuous curve shows that there are points all the way along the graph).

(vi) *As the value of x changes, the value of y also instantaneously changes,* demonstrated by the **graph** (there are no horizontal sections to the graph).

(vii) *The domain is the real numbers:* demonstrated by the **graph, table of values** and **coordinate pairs** (all x-values are real numbers).

(viii) *The range is the positive real numbers:* demonstrated by the **graph, table of values** and **coordinate pairs** (all y-values are positive real numbers).

b) Students could identify limitations in all of the representations. Some suggestions are given below:

- It is expected that most would identify the **equation** as being of limited use for explaining the properties of the function, as this abstract representation relies more on interpreting the relationship rather than seeing it.
- Students may decide the **coordinate pairs** or **table of values** are of limited use as they require some visualisation of the graph that they are representing to truly identify some properties. They also do not easily demonstrate the infinite nature of a continuous function, only the individual coordinates shown.
- Finally, the **graph** could be seen to be limiting as it is difficult to read the coordinate pairs.

c) (i) All functions (ii) Some functions (iii) Some functions (iv) Some functions

(v) Some functions (vi) Some functions (vii) Some functions (viii) Some functions

Note: some statements above refer to the features of functions referenced in the Core Concept document, and so students may argue that 'all functions' apply. However, this cannot be used if exceptions can be found. For example, statement (vi) states, 'As the value of x changes, the value of y also instantaneously changes,' which describes covariation. While this is true for any function with more than one variable, it is also true to state that $f(4)$ is a function, and so it is not a statement that applies to **all** functions.

Understand that, for the graph of any given function, a particular value of x can give only one value for y

Example 7:

Responses may vary but should demonstrate an understanding that:

a) Both graphs pass through the point (0,0). They also both have the same values in the first quadrant.

b) Billy is not correct because a function must assign exactly one output/range to each input/domain.

Example 8:

Sketches A, C, E and G

Example 9:

Points C, D, F, H and I

Example 10:

Either of (7, 7) or (7, 3) but not both.

(1, 7)

(−3, 7),

(12, 53)

(0, 0)

Appreciate that particular points on a graph offer instances of the relationship between the domain and range of a function

Example 11:

Approximately:

a) 2.25	b) 0.6	c) 1.8
d) −2.6, −1.8, 1.1	e) −3.35, 0	f) −3.475

Example 12:

a) Responses between $f(3) = 6$ and $f(2) = 4$.

Responses may vary but should indicate an understanding that:

b) Players must choose values between the previous two turns.

c) It is always possible for one of the students to pick a value between the previous two inputs so the game could go on forever.

d) For a continuous domain, it will always be possible to find a value between two given values (although it is unlikely students will use that language at this stage).

e) Bea is correct for the functions given but may not be correct for all functions.

Understand graphs as offering an insight into the general relationship between the domain and range of a function

Example 13:

Responses may vary but may demonstrate an understanding that:

a) Since it is clear to see 3 on the x-axis, version C is the best choice.

b) The shape's cubic nature is visible on both versions B and C, and easier to discern on version C.

c) Only version B shows the y-value when $x = 10$.

9.4.2.1 Understand the nature and graphical features of an exponential relationship

Relate the intersection at (0, 1) to the structure of exponential relationships

Example 1:

a)

	x		
	-1	0	1
2^x	$\frac{1}{2}$		2
3^x	$\frac{1}{3}$	1	3
4^x	$\frac{1}{4}$	1	4
11^x	$\frac{1}{11}$	1	11
17^x	$\frac{1}{17}$	1	17
a^x	$\frac{1}{a}$	1	a

b) Graphs plotted using correct coordinates from table in part a, showing the key features of exponential curves (i.e. all have a horizontal asymptote at $y = 0$) and a vertical asymptote which gets closer to the y-axis as the value of the base increases. Images of correct graphs can be sourced by typing the equations in the first column into any graphing software (many of which are freely available).

c) Responses may vary but may include noticing that each exponential function in this table intersects the y-axis at (0,1). Students may also notice that the larger the base number, the faster the rate of growth.

Example 2:

Responses may vary but should demonstrate an understanding that:

a) As the overall shape size increases, the area is doubling each time.

b) As the overall shape size decreases, the area is halving each time.

c) 1 unit, (2 units), 4 units, 8 units, 16 units.

d) 2^0, (2^1), 2^2 2^3, 2^4. The index is increasing by 1 each time from left to right.

e) $\frac{1}{4}$ unit, $\frac{1}{2}$ unit, 1 unit, (2 units), 4 units.

f) 2^{-2}, 2^{-1}, 2^0, (2^1), 2^2. The index is still increasing by 1 each time but we now have negative indices.

g) They could have chosen any of the shapes in their sequence. If the first shape represented 2 units, the powers would have gone from 2^1 to 2^5. If the third shape represented 2 units, the powers would have gone from 2^{-1} to 2^3. If the fifth shape represented 2 units, the powers would have gone from 2^{-3} to 2^1.

Relate the constantly changing rate of change to the structure of exponential relationships

Example 3:

29 days

Example 4:

a) 3 lots of stick B

b) 3 lots of stick D

c) 9 lots of stick C

d) 27 lots of stick D

e) 27 lots of stick E

Example 5:

a) Responses may vary but should demonstrate an understanding that both cross the y-axis; both have a similar shape in quadrant 1; some coordinates (3) will be equal; range is $y \geq 0$. Conversely, they have a different y-intercepts; $g(x)$ is symmetrical about the line $x = 0$ while $f(x)$ is not; $g(x)$ passes through the origin; $f(x)$ is asymptotic to the x-axis as x approaches negative infinity; $g(x)$ is a parabola.

b) Responses may vary but may include an understanding that as x increases from 0, $f(x)$ gets much larger than $g(x)$ and as x decreases from 0 the converse is true.

c)

x	−10	−9	−8	−7	−6	−5	−4	−3	...	6	7	8	9	10
f(x)	$\frac{1}{1024}$	$\frac{1}{512}$	$\frac{1}{256}$	$\frac{1}{128}$	$\frac{1}{64}$	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{8}$	...	64	128	256	512	1024
g(x)	100	81	64	49	36	25	16	9	...	36	49	64	81	100

d) Responses may vary but may include an understanding that the difference between $f(x)$ and $g(x)$ are more significant the larger the absolute value of x.

Relate the asymptote to the structure of exponential relationships

Example 6:

a) $f(x) = 1^x$ is D, $f(x) = 2^x$ is C, $f(x) = 3^x$ is D, $f(x) = 10^x$ is C.

b) Graph E could be $f(x) = n^x$ where $n > 10$. The exact answer is $f(x) = 100^x$.

9.4.3.5 Appreciate the connections between the graphical and the algebraic representation of translations of functions

Interpret the notation $f(x) + a$ and $f(x + a)$, evaluating the impact of the transformation numerically

Example 1:

a) $f(x) = 10x + 2$

x	−1	0	1	2	3
$f(x)$	−8	2	12	22	33
$f(x) + 3$	−5	5	15	25	35
$f(x + 3)$	22	32	42	52	62

b) Responses may vary but should demonstrate an understanding that we are adding three to the function's output.

c) Responses may vary but should demonstrate an understanding that because we are calculating $10(x + 3) + 2$, each value is (10×3) more than $f(x)$.

d) $f(x) = x^2 + 2$

x	−1	0	1	2	3
$f(x)$	3	2	3	6	11
$f(x) + 3$	6	5	6	9	14
$f(x + 3)$	6	11	18	27	38

e) Responses may vary but should demonstrate an understanding that Adam's statement is still true because we are still adding three to the function's output regardless of the function.

f) Responses may vary but should demonstrate an understanding that Salma's statement is no longer true. We are squaring a number that is 3 more than the original value of x.

Example 2:

a) Responses may vary but should demonstrate that as x increases by 1, $g(x)$ increases by 12.

b) (19, 302)

c) (19, 301 + 12) = (19, 313)

Interpret the notation $f(x) + a$ and $f(x + a)$, evaluating the impact of the transformation graphically

Example 3:

a) Graphs plotted using correct coordinates from table in the rubric for *Example 4*, with smooth parabolic curves joining these points. Images of correct graphs can be sourced by typing the equations in the first column into any graphing software (many of which are freely available).

b) Responses may vary but should demonstrate understanding that it is a vertical translation up, more specifically a translation of +3 on the y-axis.

c)	Responses may vary but should demonstrate understanding that it is a horizontal translation to the left, more specifically a translation of −3 on the x-axis.
d)	$y = f(x) - 5$ would be a vertical translation down, more specifically a translation of -5 on the y-axis. $y = f(x - 5)$ would be a horizontal translation to the right, more specifically a translation of +5 on the x-axis.
e)	$y = f(x) + 2$ would be a vertical translation up, more specifically a translation of +2 on the y-axis. $y = f(x + 2)$ would be a horizontal translation to the left, more specifically a translation of −2 on the x-axis.

Example 4:

a) (i) p (ii) q (iii) r

b) (−10, 0) where line p meets the x-axis; (0, 100) where line p intercepts the y-axis; (−10, 16) where line r meets line q; (0, 16) where line q intercepts the y-axis; (0, 116) where line r intercepts the y-axis.

9.5.1.3 Understand the graphical features of the sine, cosine and tangent functions with arguments in degrees

Understand it is possible to turn an angle greater than 360°

Example 1:

a) (i) Ray (ii) Graham

b)

c) They are turning clockwise, although we only know this because of Graham and Ray's final positions. Miles could have turned in either direction to end up in the same position.

Example 2:

a) Regardless of which person students select, they will appear correct as all three statements are possible. Ideally, students will recognise this.

b) Responses may vary but should demonstrate students' understanding that: for Lizi to be correct, the point has moved just 110°; for Steve to be correct, the point has completed one full turn and then the 110°; for Beth to be correct, the point has completed ten full turns and then the 110°.

Connect the graphs of trigonometric functions to the unit circle

Example 3:

a) No. It has increased by roughly one and three-quarter times.

b) No. It has doubled.

c) Responses may vary but should demonstrate an understanding that between 90° and 180° the height of P reduces as the angle increases. When the original angle is multiplied by 4, the height is the same as when the original angle was multiplied by 3. When the original angle is multiplied by 5, the height is the same as when the original angle was multiplied by 2.

d) Responses may vary but should demonstrate an understanding that:

- Between 0° and 90°, the height of P increases as the angle increases, though at a decreasing rate.
- Between 90° and 180°, the height of P decreases as the angle increases. The heights are a mirror image of each other about the y-axis.
- Between 180° and 360°, the height of P is a mirror image of the heights between 0° and 180° resulting in negative 'heights'.
- The maximum height is at 90°.
- At 180° and 360° the height is 0.

Example 4:

a) Graph X

b) Graph of $sin(x)$ plotted using correct coordinates from the table in the rubric, showing a smooth and continuous curve through all points. An image of the correct graph can be sourced by typing the equations into any graphing software (many of which are freely available).

c) Responses may vary but should demonstrate an understanding that the heights for angles 90° to 180° are the mirror image of the heights for angles 0° to 90° with the line of symmetry at $x = 90^\circ$.

Students may refer to part d of *Example 3*. The heights for angles 180° to 360° are the mirror image of the heights of 0° to 180° but they are multiplied by −1.

Example 5:

a) 1 unit (as it is the radius of a unit circle.)

b) $cos(\theta) = \frac{adj}{hyp}$ Here, $cos(30) = \frac{OQ}{1}$ *So,* $OQ = cos(30)$

c) $sin(\theta) = \frac{opp}{hyp}$ Here, $sin(30) = \frac{PQ}{1}$ *So,* $PQ = sin(30) = \frac{1}{2}$

d) The distances *OQ* and *PQ* have changed. *OP* remains as 1 unit.

e) *OQ* = $cos(\theta)$ *PQ* = $sin(\theta)$

f) $sin(\theta)$ will always be the height opposite the angle, θ, so the vertical distance between P (the point on the circumference of the unit circle) and Q (the point where this vertical line meets the x-axis).

$cos(\theta)$ will always be the side adjacent to the angle, θ, so the horizontal distance between O (the origin) and Q (the point where the vertical line from P meets the x-axis).

g) $cos(0) = 1, cos(90) = 0,\ cos(180) = -1, cos(270) = 0$

h) Correctly plotted graph, as shown in the guidance column for this example. (Note that this image shows $cos(\theta)$ between -360° and 360°, but students need to only plot the points and curve from 0° to 360°.

i) $cos(360) = 1, cos(540) = -1, cos(720) = 1$. This repeating cycle of values is because the cosine graph will continue indefinitely along the x-axis. It is periodic with a cycle of $360°$

Example 6:

a) Responses are likely to vary but students may notice the heights marked on the tangent are increasing significantly. Students may wonder how big it can get and what happens once P has moved through 90°.

b) Responses may vary but should demonstrate an understanding that point D will be much higher on the graph.

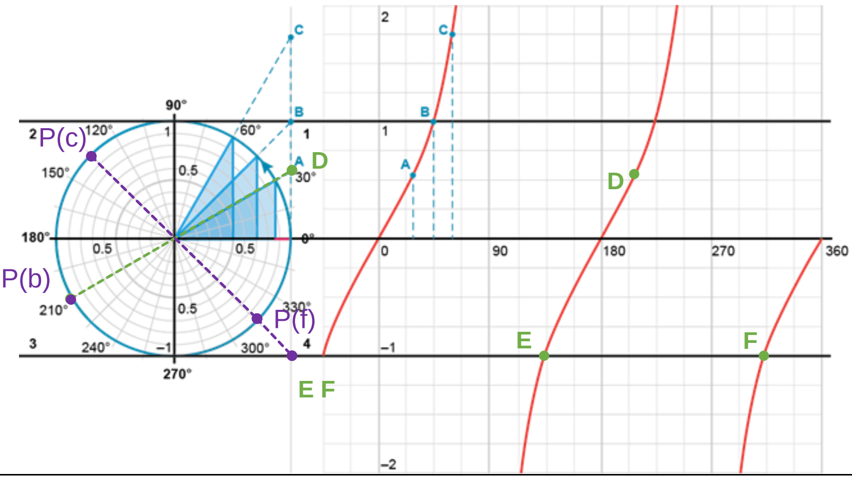
c) Responses may vary but should demonstrate an understanding that it will form a vertical line which will never meet the tangent.

Example 7:

a) Point D is marked on the graph and the unit circle in the image below.

b) The position of point P on the unit circle is also marked on the image below and labelled P(b).

c) Responses for part c will vary, and responses for parts d, e, f will follow on from the point selected in part c. An exemplar response is shown below, with the relevant positions of point P denoted by P(c) and P(f).



Understand that, for the graph of any trigonometric function, there are multiple values of x for a particular value of y

Example 8:

a) Responses may vary but should demonstrate an understanding that it is approximately 0.86.

b) Responses may vary but should demonstrate an understanding that it is approximately −0.17.

c) 210° or 330° (or $(360n + 210)°$ or $(360n + 330)°$)

d) Responses may vary but may demonstrate reasonable confidence though it is difficult to read the exact values from the graph.

e) $3 \times 360 + 330 = 1510°$

Example 9:

Responses to the blank statements in the left-hand column may vary; some suggestions are shown in the table below:

	$Sin(x)$	$Cos(x)$	$Tan(x)$
Is symmetrical between 0° and 360°		✓	
Is positive between 0° and 90°	✓	✓	✓
Is positive between 90° and 180°	✓		
Is continuous	✓	✓	
Intersects the x-axis at 180°	✓		✓
e.g. has undefined values			✓
e.g. intersects the x-axis at 180°		✓	
e.g. has a height of 1 at 90°	✓		
e.g. has a minimum y-value of -1	✓	✓	
e.g. intersects the x-axis at 0°	✓		✓
e.g. the graph is periodic	✓	✓	✓

Example 10:

a) Highest point: 90°. Lowest point: 270°.

b) It could have turned 30° or 150°.

c) (i) If the radius is 2, then the shape of the graph remains the same, but it is 'stretched' vertically so that the highest and lowest points are at 2 and -2 respectively.

(ii) If the height above the bottom of the unit circle is used, then the graph would have the same shape as Kyle's original graph but be translated vertically up by one unit, so that the highest and lowest points are at 2 and 0 respectively.

(iii) If the speed changes, the graph would be unchanged as speed is not a variable.